THE COSET METHOD FOR RUBIK'S CUBE: SOLUTION (DRAFT)

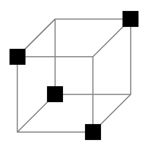
3. A solution

There are multiple approaches and multiple reasonable answers. Here's one.

3.1. **Problem 1/1'.**

3.1.1. *Conditions characterizing D.* We will give conditions satisfied by any state that is solvable using only half turns (henceforth called "half-turn state").

First, consider the following collection of corners in the solved state:



Let $S_{\text{ev}} \times S_{\text{od}}$ be the subgroup of S_8 that preserves this collection. (With the numbering from §1, this is the subgroup that permutes the even-numbered corners among themselves and the odd-numbered corners among themselves.) Since any half turn preserves this collection, we have

$$D \leq S_{\text{ev}} \times S_{\text{od}}. \tag{3.1}$$

In other words, any half-turn state satisfies

(D1): The even-numbered corners are permuted among themselves (and so the odd-numbered corners are permuted among themselves).

Next, any half turn exchanges two pairs of corners, so it is an even permutation. Hence

$$D \leq A_8, \tag{3.2}$$

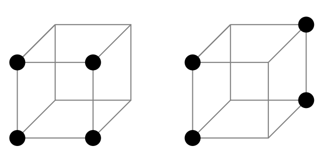
where A_8 is the alternating group (the subgroup of even permutations in S_8). In other words, any half-turn state satisfies

(D2): The corresponding permutation in S_8 has even parity.

Finally, note that any half turn sends any plane containing 4 corners to another such plane. It follows that any half-turn state satisfies

(D3): Corners $1, 2, 3, 4$ lie in a single plane.

That is, corners 1, 2, 3, 4 are in one of the following configurations, up to orienting the entire cube:



Proposition 3.1. *A state is a half-turn state if and only if it satisfies* ***(D1)–(D3)****.*

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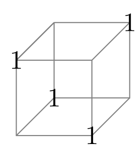
For convenience, here are the conditions again, collected:

(D1): The even-numbered corners are permuted among themselves (and so the odd-numbered corners are permuted among themselves).
(D2): The corresponding permutation in S_8 has even parity.
(D3): Corners $1, 2, 3, 4$ lie in a single plane.

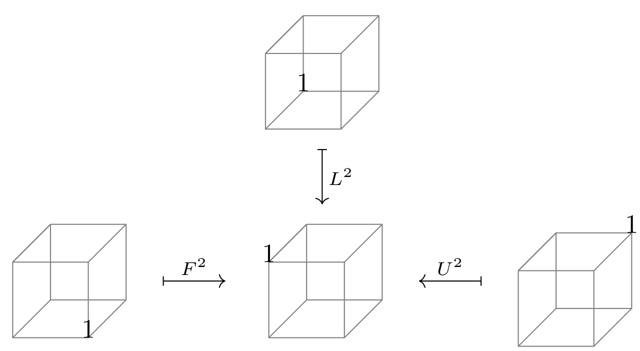
Note that, given **(D1)**, **(D2)** is quite fast to check.

Proof of Proposition 3.1. It remains to show that any state satisfying **(D1)**–**(D3)** can be solved using only half turns. We will describe such a solution. (Note that each condition is preserved under the action of D, so we may invoke these conditions at every step of the solution.)

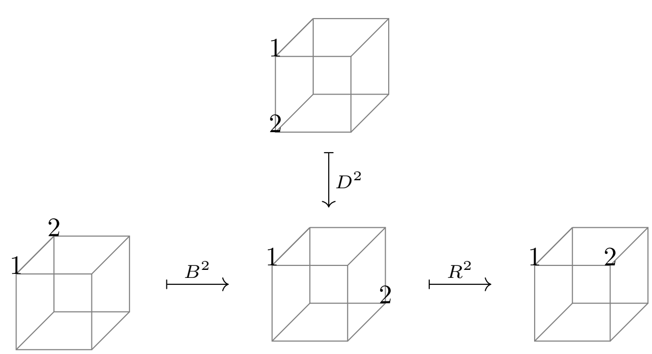
By **(D1)**, corner 1 lies in one of the following positions:



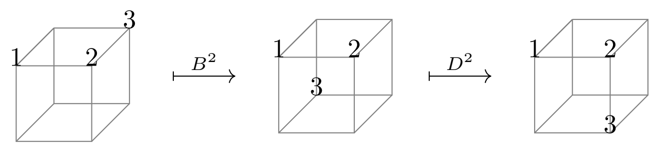
Solve it as follows:



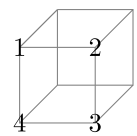
We can similarly solve corner 2:



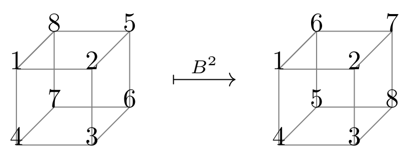
Then corner 3:



By **(D3)**, corner 4 must already be solved:



Finally, by **(D1)** and **(D2)**, corners 5, 6, 7, 8 must be in one of the following two configurations:

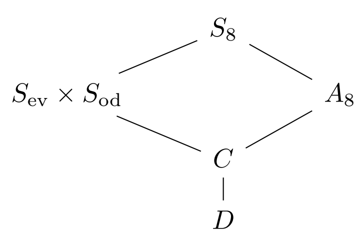


□

Let's summarize what we've done. Set

$$C = (S_{\text{ev}} \times S_{\text{od}}) \cap A_8. \tag{3.3}$$

We have the following containments of subgroups of S_8:

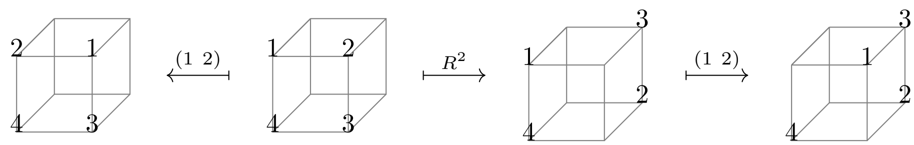


(D1) characterizes states corresponding to $S_{\text{ev}} \times S_{\text{od}}$, while **(D2)** characterizes states corresponding to A_8; requiring both gives C. By (3.2) and (3.1), we have $D \leq C$. However, this containment is strict; for example,

$$\tau = (1\ 3\ 7)$$

lies in C but not in D (i.e. $x_0.\tau$ is not a half-turn state). To characterize D, we introduced a further condition **(D3)**. Proposition 3.1 says that these three conditions suffice.

Remark 3.2. Unlike for **(D2)** and **(D1)**, the subset of S_8 corresponding to states in **(D3)** do not form a subgroup. For example, $x_0.R^2$ and $x_0.(1\ 2)$ satisfy **(D3)**, but $x_0.R^2(1\ 2)$ doesn't:

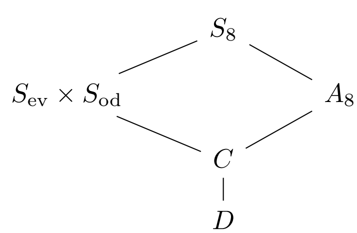


For more in this direction, see §4.

3.1.2. *Order of D.*
Approach 1. We use Proposition 3.1. By **(D3)**, corners 1, 2, 3, 4 lie in one of 12 planes. Within this plane, by **(D1)** there are 2 ways to place corners 1 and 3, and 2 ways to place corners 2 and 4. Having done this, by **(D1)** and **(D2)**, there are only 2 possible ways to position corners 5, 6, 7, 8 (as in the proof of Proposition 3.1). This gives

$$|D| = 12 \cdot 2 \cdot 2 \cdot 2 = 96.$$

Approach 2. Recall the following containments:



We know that

$$|S_{\text{ev}} \times S_{\text{od}}| = 4!4! = 576,$$

so it is enough to determine the indices $[S_{\text{ev}} \times S_{\text{od}} : C]$ and $[C : D]$.

Multiplication by a fixed 2-cycle in $S_{\text{ev}} \times S_{\text{od}}$ defines a bijection between the even permutations and the odd permutations in $S_{\text{ev}} \times S_{\text{od}}$, so

$$[S_{\text{ev}} \times S_{\text{od}} : C] = 2$$

and

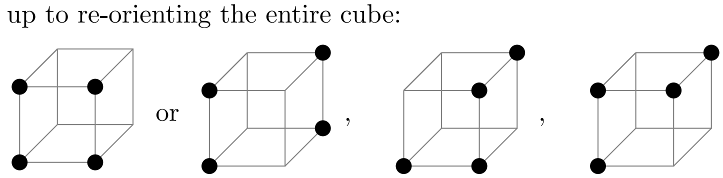
$$|C| = \frac{1}{2}|S_{\text{ev}} \times S_{\text{od}}| = \frac{1}{2}(4!4!) = 288.$$

From Approach 1, we know that $[C : D] = 288/96 = 3$. But let's give a more conceptual proof.

As above, we consider multiplication by a fixed element in C that is not in D. Recall that a 3-cycle was such an element; for ease of exposition, we use the explicit choice $\tau = (1\ 3\ 7)$. Define a map (of sets)

$$c_\tau \colon C \to \mathbb{Z}/3\mathbb{Z}$$

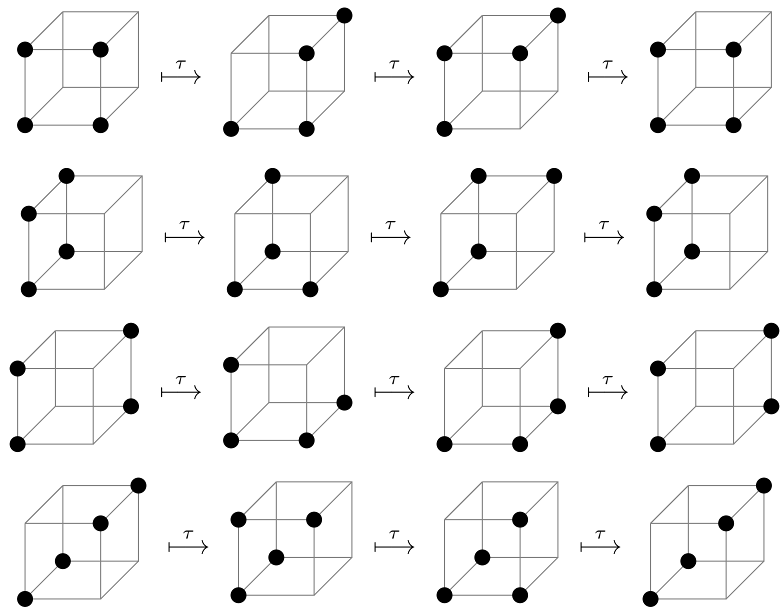
as follows. For $\sigma \in C$, the configuration of corners 1, 2, 3, 4 in $x_0.\sigma$ is one of the following, up to re-orienting the entire cube:



Define $c_\tau(\sigma)$ to be 0 in the first two, 1 in the third, and 2 in the fourth case. Then

$$c_\tau(\sigma\tau) = c_\tau(\sigma) + 1.$$

This may be seen by checking the effect of τ on each configuration; up to rotation, the following cover all cases:

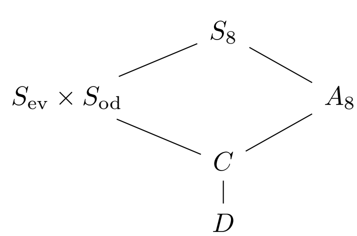


In other words, $c_\tau(\sigma) \in \{0,1,2\}$ is the unique element such that $x_0.\sigma\tau^{-i}$ satisfies **(D3)**. Since $\sigma, \tau \in C$, we know that $x_0.\sigma\tau^{-i}$ satisfies **(D1)** and **(D2)** for any $i \in \{0,1,2\}$. Hence it follows from Proposition 3.1 that $c_\sigma(\sigma) \in \{0,1,2\}$ is the unique element for which $\sigma\tau^{-i} \in D$, or equivalently, $\sigma \in D\tau^i$. We conclude that c_τ induces a bijection

$$\begin{aligned} \overline{c_\tau}\colon D\backslash C &\xrightarrow{\sim} \mathbb{Z}/3\mathbb{Z} \\ D\tau^i &\longmapsto i. \end{aligned} \tag{3.4}$$

Remark 3.3. D is not normal in C.

3.2. **Problem 2.** Recall again the following containments:



Since $S_{\text{ev}} \times S_{\text{od}}$ corresponds to states characterized by **(D1)**, the configuration of the odd-numbered corners (say) serve as a label for its right cosets:

$$\begin{aligned} (S_{\text{ev}} \times S_{\text{od}})\backslash S_8 &\xrightarrow{\sim} \{\text{config. of 4 corners}\} \\ (S_{\text{ev}} \times S_{\text{od}})\sigma &\longmapsto \text{config. of odds in } x_0.\sigma. \end{aligned} \tag{3.5}$$

The right A_8 cosets may be labeled by parity:

$$
\begin{aligned}
A_8\backslash S_8 &\xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z} \\
A_8\sigma &\longmapsto \text{parity of } \sigma.
\end{aligned} \tag{3.6}
$$

The equality (3.3) implies that the natural map

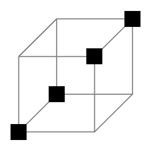
$$C\backslash S_8 \to (S_{\mathrm{ev}} \times S_{\mathrm{od}})\backslash S_8 \times A_8\backslash S_8$$

is injective; by comparing size, we see that it's even a bijection (though we won't need this). So by combining the labelings (3.5) and (3.6), we obtain a labeling of $C\backslash S_8$:

$$
\begin{aligned}
C\backslash S_8 &\xrightarrow{\sim} \{\text{config. of 4 corners}\} \times \mathbb{Z}/2\mathbb{Z} \\
C\sigma &\longmapsto (\text{config. of odds in } x_0.\sigma,\ \text{parity of } \sigma).
\end{aligned} \tag{3.7}
$$

For example,

$$C(1\ 2\ 3\ 4) \mapsto \left(\quad , 1 \right).$$



Now focus on the following containments:

$$
\begin{array}{c}
S_8 \\
| \\
C \\
| \\
D
\end{array}
$$

The idea is to combine the labelings (3.7) of $C\backslash S_8$ and (3.4) of $D\backslash C$ to produce a labeling of $D\backslash S_8$. To do this, choose a section s of the natural surjection $S_8 \twoheadrightarrow D\backslash S_8$:

$$
\begin{array}{ccc}
 & & S_8 \\
 & & \downarrow \\
D\backslash C & \hookrightarrow & D\backslash S_8 \\
 & & \downarrow \\
 & & C\backslash S_8
\end{array}
\qquad (s : C\backslash S_8 \dashrightarrow S_8)
$$

For any coset in $D\backslash S_8$, represented by say $D\sigma$ for some $\sigma \in S_8$, we have $C\sigma = Cs(C\sigma)$, so $\sigma s(C\sigma)^{-1} \in C$. We can therefore define a map

$$
\begin{aligned}
D\backslash S_8 &\to C\backslash S_8 \times D\backslash C \\
D\sigma &\mapsto (C\sigma, D\sigma s(C\sigma)^{-1}),
\end{aligned}
$$

which is easily seen to be injective, hence a bijection by comparing size. Combining (3.7) and (3.4), we obtain a labeling

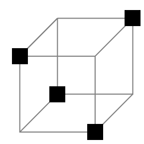
$$D\backslash S_8 \xrightarrow{\sim} \{\text{config. of 4 corners}\} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

depending on the choice of s.

Concretely, the map

$$\begin{aligned} S_8 &\longrightarrow C \\ \sigma &\longmapsto \sigma s(C\sigma)^{-1} \end{aligned}$$

assigns a state in C to any state σ in a way that only depends on $C\sigma$. In other words (in terms of the labeling (3.7)), given any state, we want to use only its configuration of odd-numbered corners and the parity of the corresponding permutation to systematically produce a state whose labels are



$$\left(\quad , 0 \right).$$

There are many ways to do this (corresponding to the choice of s), each leading to an easily-computable labeling for $D\backslash S_8$.

4. Another approach (sketch)

Recall from Remark 3.2 that **(D3)** is not a "group-like" condition. The problem was that **(D3)** favored a single plane (the one containing 1, 2, 3, 4). We can correct this as follows. Let

$$P \leq S_8$$

be the subgroup that takes any plane of 4 corners to another such plane. Then

$$D \leq P, \tag{4.1}$$

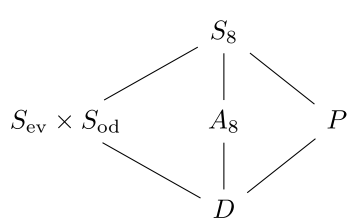
so any half-turn state satisfies the following strengthening of **(D3)**:

(D3s): Every collection of 4 corners that lie in a single plane in the solved state, lie in a single plane.

By (3.2), (3.1), and (4.1), we have

$$D \leq (S_{\text{ev}} \times S_{\text{od}}) \cap A_8 \cap P.$$

Thus we have the following containment of subgroups:



Proposition 3.1 implies that in fact

$$D = (S_{\text{ev}} \times S_{\text{od}}) \cap A_8 \cap P, \tag{4.2}$$

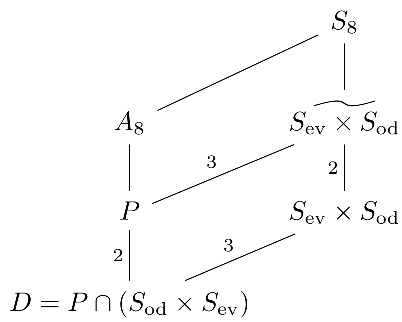
so the corresponding conditions **(D1)**+**(D2)**+**(D3s)** provide another characterization of half-turn states.

(D3s) is a rather strong condition. One can show that it implies **(D2)** (so $P \leq A_8$), and that given **(D3s)**, we only need the following weakening of **(D1)** to find a half turn solution:

(D2w): Corner 1 lies in an odd-numbered corner position.

In fact, this shows that D is an index 2 subgroup inside P, with the nontrivial coset represented for example by any 90° re-orientation of the entire cube.

This last characterization of half turns (**(D2w)**+**(D3s)**) does not involve counting parity. On the other hand, since there are 6 pairs of complement planes of 4 corners, checking **(D3s)** seems more time-consuming.



Group-like conditions have the following advantage: (4.2) implies that the natural map

$$D\backslash S_8 \to (A_8\backslash S_8) \times ((S_{ev} \times S_{od})\backslash S_8) \times (P\backslash S_8)$$

is injective. It therefore suffices to name the coset spaces $A_8\backslash S_8$, $(S_{ev} \times S_{od})\backslash S_8$, and $P\backslash S_8$ individually. However, it's not clear to me how to do this for $P\backslash S_8$ without keeping track of the configuration of 6 planes of 4 corners.

One could hope that there is a subgroup $Q \leq S_8$ with a small index inside S_8 (so that $Q\backslash S_8$ is easy to describe) so that we still have

$$D = (S_{ev} \times S_{od}) \cap A_8 \cap Q.$$