Structure Theory of Reductive Groups through Examples

Shotaro Makisumi

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Abstract

This expository paper provides an overview of the structure theory of reductive groups, first over algebraically closed ground fields and more briefly over arbitrary fields. Explicit computations for several low-rank classical groups are given to illustrate the general theory.

1 Introduction

A systematic study of linear algebraic groups was initiated by C. Chevalley and A. Borel in the mid 1950s, a few decades after the classification of complex finite-dimensional semisimple Lie algebras by W. Killing and E. Cartan. Since then, the Chevalley-Borel theory and its extensions by others have played important roles in a number of algebraic areas, from finite simple groups to arithmetic subgroups of Lie groups. For a brief historical overview, see for example the preface of [7].

This expository paper provides an overview of the structure theory of reductive groups, including more briefly the case of non-algebraically closed ground fields, with several explicit examples of low-rank classical groups. We only treat affine K-varieties, that is, reduced affine K-schemes of finite type. The exposition is intended for an advanced undergraduate with minimal background in algebraic geometry (an introductory course on varieties would suffice). Familiarity with the classification of semisimple Lie algebras is helpful, and several parallels to that theory are pointed out.

Section 3 contains an overview of the general theory over an algebraically closed field (the "absolute" theory), with few proofs and skipping several developments from standard expositions. Each subsection in Section 4 deals with one classical group ($\mathrm{GL}_3(K)$, $\mathrm{SL}_3(K)$, $\mathrm{PSL}_3(K)$, $\mathrm{Sp}_4(K)$, $SO_5(K)$) and explicitly works out the major points of the general theory as well as particular phenomena of interest, such as isogeny. After a brief discussion in Section 5 of differences to the theory that arise for arbitrary ground fields (the "relative" theory), Section 6 looks at several low-rank orthogonal groups to highlight these differences.

Throughout, K denotes an algebraically closed field and k a subfield of K. Some standard texts on this structure theory are [1], [4], [12]; more specific references are given in each section. Even for some topics that are only briefly touched on, multiple references of varying depth have been included to facilitate further study.

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2 Generalities on Linear Algebraic Groups

2.1 First Notions and Results

Concretely, a *linear algebraic group* over K is a (Zariski) closed subgroup of some $\mathrm{GL}_n(K)$. A more abstract definition free of embedding begins with a K-variety. An *algebraic group* over K is a K-variety[1] G with a compatible group structure: multiplication $G \times G \to G$ and inversion $G \to G$ are morphisms of K-varieties. A *morphism* of algebraic groups is a group homomorphism that is also a morphism of K-varieties. Write $\mathrm{Hom}(G, H)$ for the set of morphisms from G to H as algebraic groups. An *affine* or *linear* algebraic group is one whose underlying variety is affine. It is a theorem that every such algebraic group can be embedded as a closed subgroup of some $\mathrm{GL}_n(K)$, which justifies the adjective "linear" as well as the more concrete definition given above.

Example 2.1. *$\mathbb{G}_m$ is the one-dimensional multiplicative group variety $K^\times$. As a K-variety, it is an open set of $\mathbb{A}^1$ defined by $T \neq 0$, so it has coordinate ring $K[T, T^{-1}]$. The group structure is given by multiplication $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ induced from K, with inversion say $\iota : \mathbb{G}_m \to \mathbb{G}_m$. On the coordinate rings, these induce, respectively, the* comultiplication $m^* : K[T, T^{-1}] \to K[T, T^{-1}] \otimes K[T, T^{-1}]$ *given by $T \mapsto T \otimes T$ and the* antipode map $\iota^* : K[T, T^{-1}] \to K[T, T^{-1}]$ *given by $T \mapsto T^{-1}$. These induced maps are K-algebra homomorphisms, so by the contravariant equivalence of categories between affine K-varieties and reduced finitely generated K-algebras, multiplication and inversion are morphisms. Thus $\mathbb{G}_m$ is indeed a linear algebraic group. Note that $\mathbb{G}_m$ can be embedded as a closed subgroup of a general linear group, as simply* $\mathrm{GL}_1(K)$ *or in* $\mathrm{GL}_2(K)$ *via $x \mapsto \mathrm{diag}(x, x^{-1})$.*

For later use, we determine the morphisms $\varphi : \mathbb{G}_m \to \mathbb{G}_m$. Under the above contravariant equivalence of categories, each corresponds in particular to a K-algebra endomorphism $\varphi^* : K[T, T^{-1}] \to K[T, T^{-1}]$. Since $\varphi^*(T)$ and $\varphi^*(T^{-1})$ both lie in $K[T, T^{-1}]$ and multiply to 1, considering the exponents of T in the product shows that $\varphi^*(T) = aT^m$ for some $a \in K^\times$ and $m \in \mathbb{Z}$, which uniquely determine φ^*. This corresponds to $\varphi(x) = ax^m$, which gives the endomorphisms of the underlying K-variety. That φ is a group homomorphism then forces $a = 1$, so $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \{x \mapsto x^m : m \in \mathbb{Z}\}$. Alternatively, we can directly use the compatibility of φ^* with comultiplication.

Let G be a linear algebraic group. Since G is a Noetherian topological space, it is the union of finitely many irreducible components. Since G is a variety (and in particular reduced), it has

[1] In particular, a reduced K-scheme of finite type.

some simple point. Then the transitive action of G on itself by left multiplication shows that every point is simple, i.e. G is smooth.[2] This implies that the irreducible components of G agree with its connected components. The component containing the identity, called the *identity component* and denoted G°, is a finite-index normal subgroup of G. The theory therefore focuses on connected groups.[3]

2.2 Jordan Decomposition, Unipotent and Diagonalizable Groups, Torus

An element $x \in \mathrm{GL}_n(K)$ is called *semisimple* if it is diagonalizable as a matrix, nilpotent if $x^m = 0$ for some m, and *unipotent* if $x - I$ is nilpotent, where I is the identity matrix. By linear algebra, any $x \in \mathrm{GL}_n(K)$ has a unique *(multiplicative) Jordan decomposition* $x = x_s x_u$, where x_s is semisimple, x_u is unipotent, and $x_s x_u = x_u x_s$. Then x_s and x_u are respectively called the *semisimple* and *unipotent part* of x. In a linear algebraic group, there exists an analogous unique decomposition of each element x into "semisimple" part x_s and "unipotent" part x_u that is independent of embedding, i.e. such that, for any embedding φ into $\mathrm{GL}_n(K)$, we have $\varphi(x_s) = \varphi(x)_s$ and $\varphi(x_u) = \varphi(x)_u$. From the uniqueness, one easily proves the preservation of the Jordan decomposition: if $\varphi : G \to G'$ is a morphism of algebraic groups, then $\varphi(x)_s = \varphi(x_s)$ and $\varphi(x)_u = \varphi(x_u)$.

An element x is *semisimple* if $x = x_s$ and *unipotent* if $x = x_u$. For a linear algebraic group G, the subset of semisimple (resp. unipotent) elements is denoted G_s (resp. G_u). A group G is *unipotent* if $G = G_u$. A group is *diagonalizable* if it is commutative and consists of semisimple elements, or equivalently (since a commuting set of diagonalizible matrices can be simultaneously diagonalized), if it is isomorphic to a closed subgroup of some diagonal group $D_n(K) \cong \mathbb{G}_m^n$. A *torus* is a connected diagonalizable group, or equivalently, a group isomorphic to some $\mathbb{G}_m^n$.

2.3 Reductive and Semisimple Groups

Any linear algebraic group G has a unique largest normal solvable subgroup, which is then automatically closed. Its identity component is the largest connected normal solvable subgroup, called the *radical* of G and denoted $R(G)$. The unipotent part of $R(G)$ is the largest connected normal unipotent subgroup of G, called the *unipotent radical* and denoted $R_u(G)$.

A connected group G is *semisimple* if $R(G)$ is trivial and *reductive* if $R_u(G)$ is trivial. A group is thus semisimple if it has no non-trivial connected normal solvable subgroup, or equivalently (by considering the central series) no non-trivial connected normal abelian subgroup. For example, $\mathrm{SL}_n(K)$ is semisimple while $\mathrm{GL}_n(K)$ is only reductive. If G is connected, then $G/R(G)$ is semisimple and $G/R_u(G)$ is reductive. This breaks the study of connected linear algebraic groups into that of, for example, solvable groups, reductive groups, and their extensions.

The structure theory below applies to reductive linear algebraic groups (or simply called *reductive groups*). The reader is encouraged to read the general theory in Section 3 together with the

[2] In a more general definition of algebraic groups that allows non-reduced schemes, it can be shown that an algebraic group is smooth if and only if it is reduced; see [7].

[3] For linear algebraic groups, the adjective "connected" is preferred over "irreducible" because the latter also applies to representations.

explicit examples in Section 4.

3 Structure Theory over Algebraically Closed Fields

3.1 Maximal Torus, Characters and Cocharacters, Roots, Weyl Group

From here on, G will denote a reductive group. As with Lie algebras, the structure theory of reductive groups uses root systems arising from adjoint action. The role of Cartan subalgebra for reductive Lie algebras will be played here by a *maximal torus*, a torus in G contained in no other torus. Such a maximal torus exists by dimensional reason, and it can be shown that all maximal tori are conjugate. The dimension of maximal tori is therefore well defined, called the *rank* of G. We fix a maximal torus T throughout.

Let $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$, called the *character module* and *cocharacter module*, respectively. Since T is fixed, we often write X^* and X_*. An element of X^* (resp. X_*), called a *character* (resp. *cocharacter*, *one-parameter subgroup*, or *1-psg*) of T, is often denoted α (resp. λ). Because $T \cong \mathbb{G}_m^n$ for some n, X^* and X_* are free abelian of rank n. Composing a cocharacter $\lambda : \mathbb{G}_m \to T$ and a character $\alpha : T \to \mathbb{G}_m$ yields a morphism $\mathbb{G}_m \to \mathbb{G}_m$, which, as we saw in Example 2.1, must have the form $x \mapsto x^m$ for some $m \in \mathbb{Z}$. We may therefore define $\langle\ ,\ \rangle : X^* \times X_* \to \mathbb{Z}$ by $(\alpha \circ \lambda)(x) = x^{\langle \alpha, \lambda \rangle}$. It is easy to see that this is a dual pairing of X^* and X_*.

Consider the adjoint action (i.e. conjugation) of T on $\mathfrak{g}$. By the preservation of the Jordan decomposition, $\mathrm{Ad}\, T$ is diagonalizable. We therefore have a decomposition $\mathfrak{g} = \bigoplus_{\alpha \in X^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the space of $X \in \mathfrak{g}$ with $tX = \alpha(t)X$ for all $t \in T$. A *root* is a nonzero character α for which $\mathfrak{g}_\alpha \neq 0$; then $\mathfrak{g}_\alpha$ is a *root space*. The set of roots is denoted Φ. The fixed-point space, corresponding to the zero character, is the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ of T, so $\mathfrak{g} = \mathfrak{c}_{\mathfrak{g}}(T) \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$. It can be shown that $\dim \mathfrak{g}_\alpha = 1$ for $\alpha \in \Phi$.

The *Weyl group* of G (with respect to T) is defined to be $W = N_G(T)/C_G(T)$, the normalizer of the torus modulo its centralizer. It can be shown that this is a finite group. For a maximal torus T of a reductive group G, it can be shown that $C_G(T) = T$. The reader may therefore encounter the definition of the Weyl group in this setting as simply $N_G(T)/T$. By the conjugacy of maximal tori, the Weyl group of a linear algebraic group is determined up to isomorphism. For $\sigma \in W$, represented by say $n \in N_G(T)$, conjugation by n is a morphism $T \to T$ depending only on $\sigma \in W$. Given a character $\alpha : T \to \mathbb{G}_m$, let $\sigma\alpha$ be the character obtained by composing with this conjugation: $(\sigma\alpha)(t) = \alpha(n^{-1}tn)$. This defines an action of W on X^*, which is easily seen to permute Φ. Similarly, W acts on X_* by $(\sigma\lambda)(x) = n\lambda(x)n^{-1}$. Note the direction of conjugation, which ensures that $\langle \sigma\alpha, \sigma\lambda \rangle = \langle \alpha, \lambda \rangle$ for $\sigma \in W$, $\alpha \in X^*$, $\lambda \in X_*$.

3.2 Root System

Identifying X^*, X_* with lattices in $\mathbb{R} \otimes X^*$, $\mathbb{R} \otimes X_*$, respectively, we view Φ as a subset of $\mathbb{R} \otimes X^*$. The dual pairing of X^* and X_* extends to that of $\mathbb{R} \otimes X^*$ and $\mathbb{R} \otimes X_*$, denoted again by $\langle\ ,\ \rangle$. As well, the action of W on X^* and X_* extends to a linear action on $\mathbb{R} \otimes X^*$ and $\mathbb{R} \otimes X_*$, respectively. A major result of the structure theory over algebraically closed fields is the following: if G is semisimple, then Φ is a reduced root system in $\mathbb{R} \otimes X^*$ with rank that of G and Weyl group isomorphic to W. More generally, if G is reductive, then its roots Φ can be identified with those of its derived group (G, G), which is semisimple, and Φ is a root system in the subspace of $\mathbb{R} \otimes X^*$ it spans; see Section 3.5 for a more precise relation between the combinatorial data of G and (G, G). Here, we briefly recall without proof the definitions and basic properties of a root system and associated combinatorial data; for details, see Section 6.1 of [2].

Let E be a finite-dimensional real vector space. A *refection* relative to $\alpha \in E$, $\alpha \neq 0$, is a linear transformation of E sending $\alpha \mapsto -\alpha$ and fixing pointwise a subspace of codimension one. Note that, unlike an orthogonal (i.e. Euclidean) reflection, this is not uniquely determined by α. A *(abstract) root system* in E is a subset Ψ satisfying: (1) Ψ is finite, spans E, and $0 \notin \Psi$; (2) if $\alpha \in \Psi$, there exists a reflection r_α relative to α leaving Ψ stable; (3) for $\alpha, \beta \in \Psi$, $r_\alpha(\beta) - \beta \in \mathbb{Z}\alpha$. Since Ψ spans E, a reflection r_α satisfying (2) is uniquely determined, so that the integrality condition (3) makes sense. A root system Φ is called *reduced*[4] if, for every $\alpha \in \Psi$, we have $c\alpha \in \Psi$ if and only if $c = \pm 1$. The dimension of E is called the *rank* of Ψ. Let E^* be the dual space of E, with dual pairing $\langle\ ,\ \rangle$. If $\alpha \in \Psi$, there exists a unique $\alpha^\vee \in E^*$ such that $r_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ for $x \in E$, which in particular implies the identity $\langle \alpha, \alpha^\vee \rangle = 2$. The subset $\Psi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ of E^* is the set of *coroots*. The *(abstract) Weyl group* of Ψ is the group generated by r_α, $\alpha \in \Psi$. It can be shown that the Weyl group is finite, so that we may average any inner product on E to obtain one invariant under W. It can be shown that this makes r_α into orthogonal reflections. A root system Ψ is called *irreducible* if it cannot be written as a proper disjoint union $\Psi = \Psi_1 \cup \Psi_2$ such that every root in Ψ_1 is orthogonal to every root in Ψ_2; irreducibility is in fact independent of the inner product chosen. A *base* of Ψ is a subset $\Delta = \{\alpha_i\}$ that is a basis of E and such that every $\alpha \in \Psi$ can be written $\alpha = \sum c_i \alpha_i$ for integers c_i of the same sign. The elements of a base are called *simple roots*. Roots α for which $c_i \geq 0$ (resp. $c_i \leq 0$) comprise the set of *positive roots* (resp. *negative roots*), denoted Ψ^+ (resp. Ψ^-).

For a reductive group G, $\mathbb{R} \otimes X^*$ and $\mathbb{R} \otimes X_*$ with the extended dual pairing play the role of E and E^*, respectively. For $\alpha \in \Phi$, it can be shown that a certain element σ_α of its Weyl group W act as a reflection relative to α, and that the coroots $\Phi^\vee$ in fact lie in X_*. It is often convenient to take as above inner products on $\mathbb{R} \otimes X^*$ and $\mathbb{R} \otimes X_*$ that are invariant under W, called *admissible* inner products, which then make σ_α, $\alpha \in W$, into orthogonal reflections.

We return to the classification of semisimple groups in Section 3.4.

3.3 Spherical Apartment, Parabolic Subgroups, and Borel Sugroups

A *Borel subgroup* of a linear algebraic group is a maximal connected solvable subgroup, which exists by dimensional reason. The set of Borel subgroups containing a torus T is denoted $\mathfrak{B}^T$. A *parabolic subgroup* is a subgroup P such that G/P is a projective variety. Standard expositions

[4]Although some authors include this in the definition of a root system, we allow non-reduced root systems since they arise in the relative theory; see Section 5.2

introduce these subgroups early and use them throughout the structure theory. Here we only note the following major results: all Borel subgroups are conjugate, and Borel subgroups are precisely the minimal parabolic subgroups. Note that $R(G)$ is the identity component of the intersection of all Borel subgroups; this is a connected normal solvable subgroup, and every such group is contained in this subgroup by the conjugacy of Borel subgroups. We introduce these subgroups using the spherical apartment.[5]

The *spherical apartment* of G is $\mathbb{R}\otimes X_*$ equipped with hyperplanes $H_\alpha = \{\lambda \in \mathbb{R}\otimes X_* : \langle \alpha, \lambda \rangle = 0\}$ for $\alpha \in \Phi$. A cocharacter $\lambda \in \mathbb{R} \otimes X_*$ is called *regular* if $\langle \alpha, \lambda \rangle \neq 0$ for all $\alpha \in \Phi$, i.e. if it does not lie on any H_α. The connected components of the complement of $\bigcup H_\alpha$ are called *Weyl chambers*. Thus regular cocharacters are precisely those lying in some Weyl chamber.

Note that each hyperplane H_α partitions $\mathbb{R} \otimes X_*$ into three parts depending on the sign of $\langle \alpha, \lambda \rangle$: zero on H_α itself, and positive or negative on each side. The cocharacters lying in a given Weyl chamber is characterized by these signs, either positive or negative, for the various roots. More generally, the spherical apartment is partitioned into *facets*, maximal subsets of cocharacters having the same sign for every root. In particular, the origin is a facet of dimension zero, being the only cocharacter lying on every hyperplane, while Weyl chambers are the facets of maximal dimension.

For each root α, it can be shown that there exists a morphism $u_\alpha : \mathbb{G}_a \to G$ such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for all $t \in T$, and that this uniquely determines the image, denoted U_α, which is a one-dimensional unipotent subgroup. Moreover, U_α is the unipotent part of a Borel subgroup of G_α containing T. To every $\lambda \in \mathbb{R} \otimes X_*(T)$, associate a subgroup

$$P_\lambda = \langle T, U_\alpha : \alpha \in \Phi, \langle \alpha, \lambda \rangle \geq 0 \rangle$$

of G, which is in fact parabolic. This depends only on the facet F containing λ, so we may define P_F. If facet F' lies in the boundary of $\overline{F}$, then $\langle \alpha, F' \rangle \geq 0$ whenever $\langle \alpha, F \rangle \geq 0$, so $P_F \subset P_{F'}$. It can be shown that all parabolic subgroups containing T arise in this way. Thus the spherical apartment neatly captures the containment relations among these parabolics. In particular, Borel subgroups are the minimal parabolic subgroups, and $P_0 = \langle T, U_\alpha : \alpha \in \Phi \rangle = G$ is the maximal parabolic subgroup.

Let λ be a regular cocharacter, and set $\Phi^+ = \{\alpha \in \Phi : \langle \alpha, \lambda \rangle > 0\}$ and $\Phi^- = \{\alpha \in \Phi : \langle \alpha, \lambda \rangle < 0\}$. This determines a unique subset $\Delta \subset \Phi^+$ that is a base of Φ and with respect to which Φ^+ and Φ^- are the positive and negative roots, respectively. Note that Φ^+ and hence Δ depend only on the Weyl chamber containing λ. Conversely, starting with a base Δ, it can be shown that there exists a unique Weyl chamber such that $\langle \alpha, \lambda \rangle > 0$ for every $\alpha \in \Delta$ and λ in the Weyl chamber, which gives the inverse correspondence. We thus have bijections between $\mathfrak{B}^T$, the Weyl chambers, and the choice of positive roots or equivalently of a base of Φ.

Let $B \in \mathfrak{B}^T$ be given, and suppose B corresponds to a Weyl chamber WC. Then $-WC$ is another Weyl chamber; let $B^- \in \mathfrak{B}^T$ be the corresponding Borel subgroup. This is the unique Borel subgroup such that $B \cap B^- = T$, called the *opposite Borel subgroup* of B.

[5] For a gentle introduction to spherical and affine apartments in the context of p-adic Chevalley groups, see [9].

For semisimple groups, Φ spans $\mathbb{R} \otimes X^*$ and $\Phi^\vee$ spans $\mathbb{R} \otimes X_*$. This is no longer true for general reductive groups. All the geometric information of the spherical apartment is retained in the *reduced spherical apartment*, which is the same data intersected with the span of $\Phi^\vee$ in $\mathbb{R} \otimes X_*$.

Because of the bijection between the set of Weyl chambers and $\mathfrak{B}^T$, W also acts simple transitively (by conjugation) on $\mathfrak{B}^T$.

3.4 Isogenies, Fundamental Group, and Classification of Semisimple Groups

The classification of semisimple groups is analogous to that of semisimple Lie algebras. Recall that every (reduced) root system is a direct sum of (reduced) irreducible root systems, of which there are four infinite families and five exceptional types. Here we introduce the fundamental group. These two data classify semisimple groups.

An *isogeny* is a surjective homomorphism with finite kernel. If $G \to H$ is an isogeny, then we say that G is *isogenous* to H.[6] An algebraic group is called *almost simple* if it is non-abelian and has no non-trivial closed connected normal subgroup.[7]

For G semisimple, let G_i be the minimal closed connected normal subgroups of positive dimension. Then it can be shown that G_i are almost simple, and that $G = G_1 \cdots G_n$ with the product morphism $G_1 \times \cdots \times G_n \to G$ an isogeny. Moreover, these G_i correspond to the irreducible components of the root system Φ of G. Thus G is almost simple if and only if Φ is irreducible. In this case, on both $\mathbb{R} \otimes X^*$ and $\mathbb{R} \otimes X_*$, the admissible inner product is unique up to scalar multiplication. In particular, the angles between and relative lengths of roots and coroots are unambiguously defined, so figures of Φ and of the spherical apartment can be canonically associated with G.

The *root lattice* Q is the subgroup of X^* generated by Φ, often viewed as a lattice in $\mathbb{R} \otimes X^*$. The *weight lattice* (or *lattice of abstract weights*) is $P = \{x \in \mathbb{R} \otimes X^* : \langle x, \Phi^\vee \rangle \subset \mathbb{Z}\}$. Note that the root system alone determines Q and P; the *fundamental group* of Φ is P/Q. For G semisimple, Q is a full-rank lattice in $\mathbb{R} \otimes X^*$, so P/Q is finite. It can be shown (for example using representation theory) that we have the containment $Q \subset X^* \subset P$, leaving only finitely many possibilities for X^*. The *fundamental group* of G is $\pi(G) = P/X^*$. A semisimple group G is called *simply-connected* if $X^* = P$ (i.e. $\pi(G) = 1$) and *adjoint* if $X^* = Q$. The classification theorem gives a bijection between the set of isomorphism classes of almost simple linear algebraic groups and the set of possible irreducible root systems Φ together with a possible fundamental group, viewed as a quotient of the fundamental group of Φ.

Except for D_n, $n \geq 6$ even, a direct computation shows that the fundamental group of an irreducible root system is cyclic, so Φ and the order of $\pi(G)$ suffice to determine G. For D_n, $n \geq 6$ even, we have $P/Q \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so that for each such n, there are two non-isomorphic almost simple groups with root system D_n and fundamental group of order 2. For more details on this classification, see Sections 32 and 33 of [4], which outline the construction of an algebraic group isomorphism given an isomorphism of the classifying data, as well as the existence of groups

[6] Note that this is not a symmetric relation. For abelian varieties, which we are not concerned with here, isogeny does turn out to be symmetric.

[7] Some authors use the term *simple*. It can be shown that if G is almost simple, then $G/Z(G)$ is a simple group.

corresponding to certain data. Chevalley gave a construction for semisimple groups of adjoint type over an arbitrary field[8], which was generalized by Steinberg to arbitrary fundamental group; for details of either construction, see Part VII of [3].

Isogenous groups have the same root system but different character modules, hence different fundamental groups. Conversely, all semisimple groups with a given root system may be obtained by isogenies from the simply-connected form, as quotients of the simply-connected form by various central subgroups. In particular, for any semisimple group G, there exists a simply-connected form G' and an adjoint form G'' with the same root system as G and with isogenies $G' \to G \to G''$.

3.5 Reductive Groups and their Derived Group

Let G be reductive and T a maximal torus of G. It can be shown from the structure theory that $G = (G,G)R(G)$, so that the derived group (G,G) is semisimple. Moreover, $R(G) = (Z(G))^\circ$ is a central torus with a finite intersection with (G,G). A reductive group thus differs from its semisimple derived group only in a nontrivial central torus. In fact, it can be shown that $Z(G) = \bigcap_{\alpha\in\Phi} \ker\alpha$, and from this one can relate their structures as follows.

For $A \subset X^*$, let

$$A^\perp = \{\lambda \in X_* : \langle \alpha, \lambda\rangle = 0 \text{ for all } \alpha \in A\}$$
$$\tilde{A} = \{\alpha \in X^* : \mathbb{Z}x \cap A \neq \{0\}\}.$$

Then (G,G) has the following maximal torus and character and cocharacter modules:

$$T' = \langle \operatorname{Im}\alpha^\vee : \alpha \in \Phi\rangle$$
$$X^*(T') \cong X^*(T)/(Q^\vee)^\perp, \quad X_*(T') \cong \widetilde{(Q^\vee)}.$$

Since $\mathbb{R}\otimes Q$ is the orthogonal complement of $\mathbb{R}\otimes(Q^\vee)^\perp$, the first isomorphism maps Φ injectively into $X^*(T')$. We may therefore identify Φ with the root system Φ' of (G,G) (when restricted to their span). For further details and proofs, see Sections 8.1.6-9 of [12]. See Section 4.2 below for the example of $\mathrm{SL}_3(K)$, the derived group of $\mathrm{GL}_3(K)$.

3.6 Root Datum and Classification of Reductive Groups

Since G and (G,G) have the same root system, they also have the same coroot system and reduced spherical apartment. In particular, these data are not enough to classify reductive groups. Note that, whereas a root system spans the ambient space, the root lattice Q may not be full-rank in X^* for a general reductive group. This motivates one to look at not only the span of Φ but all of $\mathbb{R}\otimes X^*$. The *root datum* of a reductive group G is the quadruple $(X^*, X_*, \Phi, \Phi^\vee)$ with the dual pairing $X^* \times X_* \to \mathbb{Z}$, and bijection $\Phi \to \Phi^\vee, \alpha \mapsto \check{\alpha}$. It can be shown that this satisfies the axioms of an *abstract root datum*, which generalizes root systems, again with Weyl group that of G with respect to T; see Section 7.4 of [12]. It is a theorem of Chevalley that reductive groups over algebraically closed fields are classified by their root datum.

[8] in fact group scheme over $\mathbb{Z}$

3.7 An Outline of Standard Expositions

This section outlines a more standard exposition and ties up some loose ends. Some additional algebraic geometry will be assumed (in particular, the completeness of projective varieties and basic properties of complete varieties).

3.7.1 Commutative Groups and Connected Solvable Groups

From the Jordan decomposition, the following structure theorem of commutative linear algebraic groups quickly follows: if G is connected and commutative, then G_s, G_u are closed connected subgroups, and $G \cong G_s \times G_u$. After establishing results on diagonalizable groups and tori, the structure theory proceeds to nilpotent and solvable groups. The main result here is the following: if G is connected and solvable, then G_u is closed connected normal subgroup of G that contains (G, G), maximal tori of G are conjugate under $G^\infty := \bigcap \mathcal{C}^i G$ (where $\mathcal{C}^i G$ is the descending central series of G), and $G = T \ltimes G_u$ for any maximal torus T. Moreover, G_s is a subgroup if and only if G is nilpotent if and only if there is a unique maximal torus, in which case $G \cong G_s \times G_u$ and $G_s = T$.

Another important result about solvable groups is the Lie-Kolchin Theorem, which states that a nonempty connected solvable subgroup of $\mathrm{GL}(V)$ has a common eigenvector in V, or equivalently that it can be triangularized. This is the analogue of Lie's Theorem for Lie algebras.

3.7.2 Borel and Parabolic Subgroups Containing a Maximal Torus

Recall that Borel subgroups are maximal connected solvable subgroups and that parabolic subgroups are closed subgroups P such that G/P is projective. Borel subgroups play an important role in standard expositions, connecting the structure theory of solvable groups above to that of reductive groups.

The key algebro-geometric input here the Borel fixed point theorem, which states that the action of a connected solvable algebraic group on a complete variety has a fixed point. The main results about Borel and parabolic subgroups are as follows: all Borel subgroups are conjugate, as are all maximal tori; a closed subgroup P is parabolic if and only if G/P is complete if and only if P contains a Borel subgroup; the union of all Borel subgroups is G; parabolic subgroups are self-normalizing, i.e. $N_G(P) = P$.

In particular, for any fixed Borel subgroup B, we may identify G/B with the set $\mathfrak{B}$ of all Borel subgroups, as follows. By the conjugacy theorem, G acts transitively on $\mathfrak{B}$ by conjugation. Consider the orbit map $G \to \mathfrak{B}$ given by $x \mapsto xBx^{-1}$. Since B is self-normalizing, i.e. its stabilizer is B itself, we have a bijection $\varphi : G/B \to \mathfrak{B}$ given by $xB \mapsto xBx^{-1}$. Moreover, since $yxB \mapsto y(xBx^{-1})y^{-1}$, the natural action of G on G/B by left multiplication makes φ equivariant.

3.7.3 Centralizers of Tori and Roots of a Reductive Group

This subsection outlines the argument for the heart of the structure theory. It may be skipped on the first reading.

Recall that the adjoint action of T yields a decomposition

$$\mathfrak{g} = \mathfrak{c}_{\mathfrak{g}}(T) \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha). \tag{1}$$

Let $I(T)$ be the identity component of the intersection of all Borel subgroup containing T. We may similarly write

$$\mathfrak{g} = \mathcal{L}(I(T)) \oplus (\bigoplus_{\alpha \in \Psi} \mathfrak{g}'_\alpha), \tag{2}$$

where $\mathcal{L}$ denotes Lie algebra. It can be shown that $C_G(T) \subset I(T)$, so that $\Psi \subset \Phi$.

By using $I(T)$, this second decomposition allows one to appeal to earlier results of solvable groups and Borel subgroups. Important as well are the following categorization of tori: a torus S is *regular*[9] if it lies in finitely many Borel subgroups, and *singular* otherwise. It is shown that a torus S is regular if and only if $C_G(S)$ is solvable, again opening the use of results on Borel subgroups. A cocharacter λ is *regular* if its image $\lambda(\mathbb{G}_m) \subset T$ is regular; this is shown to agree with the definition previously given.

Let T be a maximal torus of G. Let $\mathfrak{B}^T$ be the set of Borel ubgroups containing T. By the conjugacy theorem, $N_G(T)$ acts on $\mathfrak{B}^T$ by conjugation. It can be shown that $C_G(T)$ lies in every such Borel, so acts trivially on $\mathfrak{B}^T$. We therefore have a well-defined action of W on $\mathfrak{B}^T$. This is in fact simple transitive, so that $|\mathfrak{B}^T| = |W|$, which is finite. Thus T is regular.

These results are used to first examine the structure of reductive groups G of semisimple rank 1, meaning that $G/R(G)$ is semisimple of rank 1. First, it is shown that $|W| = 2$. Since semisimple rank 1 easily forces $|W| \leq 2$. The other inequality takes more work and relies on a geometric lemma. Let T be a torus in $\mathrm{GL}(V)$, with a cocharacter $\lambda : \mathbb{G}_m \to T$. Then T acts on $\mathbb{P}(V)$, as does $\mathbb{G}_m$, via λ, and it is shown that these two actions have the same fixed points if λ is semiregular. By using the completeness of $\mathbb{P}(V)$ to extend the action of $\mathbb{G}_m$, one finds two distinct fixed points of $\lambda(\mathbb{G}_m)$, hence of T. This lemma is applied to the action of a maximal torus T on G/B: using the completeness of the latter to embed it in some $\mathbb{P}(V)$, we obtain that T has two fixed points of G/B and so $|W| = |\mathfrak{B}^T| \geq 2$. Hence $|W| = 2$. Much of the structure of G can be shown from here. In particular, if B and B' are the two Borels containing T, then $B \cap B' = T$.

For a general reductive group G, let $T_\alpha = (\ker \alpha)^\circ$ for $\alpha \in \Psi$. Then its centralizer $G_\alpha = C_G(T_\alpha)$, with maximal torus T, is shown to be reductive of semisimple rank 1. By the structure theory for such groups, G_α has roots $\{\pm\alpha\}$ with Weyl group of order two generated by r_α sending $\alpha \mapsto -\alpha$. Putting together the various G_α yields information about Ψ in the second decomposition. Finally, it is shown that $I(T) = T$ for G reductive, so that $\Phi = \Psi$ and the two decompositions coincide. In

[9] Some authors use the term *semiregular*, but the definitions agree for reductive groups.

particular, this shows that $\dim \mathfrak{g}_\alpha = 1$ and that $\Phi = -\Phi$. By the natural inclusion $W(G_\alpha, T) \hookrightarrow W$, we have $r_\alpha \in W$. It is this element that acts on $\mathbb{R} \otimes X^*$ as a reflection relative to α. A proof of the integrality axiom completes the proof that Φ is a root system.

3.7.4 Bruhat Decomposition

For a fixed Borel B, as usual G is a disjoint union of its double cosets. What it surprising is that representatives of the Weyl group form a set of representatives for these double cosets. This is the Bruhat decomposition: $G = \coprod_{w \in W} BwB$. Tits later reformulated the proof axiomatically by introducing the notion of a *BN-pair* (or a *Tits system*), which unified the proof for .... This notion has been generalized to give other purely combinatorial structure theories. For example, the discover of a certain BN-pair in p-adic semisimple groups by Iwahori and Matsumoto led to the Bruhat-Tits theory for p-adic reductive groups. For an introduction to this theory, see [9].

4 Examples: Classical Groups over Algebraically Closed Fields

Each subsection here discusses one classical group, working out explicitly the major points of the absolutely structure theory above. Although we treat low-rank examples, most of the discussion easily generalizes to arbitrary rank.

4.1 $\mathrm{GL}_3(K)$

The *general linear group* $\mathrm{GL}_n(K)$ is the group of invertible $n \times n$ matrices with entries in K, or more abstractly the automorphism group of a fixed n-dimensionalvector space over K. Since $\mathrm{GL}_n(K)$ is a Zariski-open subset of the n^2-dimensional affine space $M_n(K)$ of all $n \times n$ matrices, its Lie algebra $\mathfrak{gl}_n(K)$ can be identified, after translating I to the zero matrix, with $M_n(K)$.

Where similar computations occur in subsequent examples, they are carried out here for $\mathrm{GL}_3(K)$ in greater detail.

4.1.1 Maximal Torus, Character and Cocharacter Modules, Roots

We first show that $\mathrm{GL}_3(K)$ is reductive. Note that the subgroup of upper triangular matrices is a Borel subgroup; it is connected and solvable, and of maximal dimension since any connected solvable group can be triangularized by Lie-Kolchin. Conjugation by the permutation matrix corresponding to (1 3) takes this to the lower triangular matrices, so $R(\mathrm{GL}_3(K))$ lies in their intersection, the diagonal matrices $D_3(K)$. Normality easily forces the diagonal entries to be equal, so $R(\mathrm{GL}_3(K))$ is contained in the subgroup of scalar matrices, and in fact equals because the latter is a closed connected solvable normal subgroup. Then $R_u(\mathrm{GL}_3(K)) = \{I\}$, as desired.

$\mathrm{GL}_3(K)$ has rank 3: the subgroup $D_3(K) \cong \mathbb{G}_m^3$ of diagonal matrices is a torus, and in fact maximal. Fix this maximal torus

$$T = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}.$$

Here and throughout, we specify subgroups with asterisks for nonzero entries, implicitly intersected with the group. For $t \in T$, we write t_i for the (i,i)-th entry.

Recall that $C_G(T) = T$ for any maximal torus T of a reductive group G. Here, this is easy to check. Clearly $T \subset C_G(T)$ since T is commutative. Suppose $a \in C_G(T)$, so $tat^{-1} = a$ for all $t \in T$. Equating the (i,j)-th entry, we have $t_i t_j^{-1} a_{ij} = a_{ij}$ for all $t_i, t_j \in K^\times$, so $a_{ij} = 0$ if $i \neq j$, which shows the opposite inclusion.

Define morphisms $\alpha_i : T \to \mathbb{G}_m$ by $\alpha_i(t) = t_i$, i.e.

$$\alpha_1(\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}) = t_1, \quad \alpha_2(\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}) = t_2, \quad \alpha_3(\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}) = t_3.$$

Define $\lambda_i : \mathbb{G}_m \to T$, where $\lambda_i(x)$ is diagonal with x in the (i,i)-th entry and 1 elsewhere, i.e.

$$\lambda_1(x) = \begin{pmatrix} x & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \lambda_2(x) = \begin{pmatrix} 1 & & \\ & x & \\ & & 1 \end{pmatrix}, \quad \lambda_3(x) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & x \end{pmatrix}.$$

Then X^* (resp. X_*) is free abelian on α_i (resp. λ_i), with dual pairing $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$, the Kronecker delta.

Write for example (a_{ij}) for the matrix with (i,j)-th entry a_{ij}. The adjoint action of T on $\mathfrak{g}$ is

$$\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} (a_{ij}) \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}^{-1} = (t_i t_j^{-1} a_{ij}),$$

so the generalized eigenvectors are E_{ij}, the matrix with 1 in (i,j)-th entry and 0 elsewhere, with character $\alpha_i - \alpha_j$:

$$tE_{ij}t^{-1} = t_i t_j^{-1} E_{ij} = (\alpha_i - \alpha_j)(t) E_{ij}.$$

The roots are the nonzero characters, so we have $\Phi = \{\alpha_i - \alpha_j : i \neq j\}$, with one-dimensional root spaces $\mathfrak{g}_{\alpha_i - \alpha_j} = \langle E_{ij} \rangle$. The corresponding one-dimensional unipotent subgroups are obtained by exponentiating the root spaces. For example, since

$$(xE_{12})^2 = \begin{pmatrix} 0 & x & \\ & 0 & \\ & & 0 \end{pmatrix}^2 = 0 \quad \Rightarrow \quad \exp(xE_{12}) = I + xE_{12} = \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix},$$

we set

$$\begin{aligned} u_{\alpha_1 - \alpha_2} : \mathbb{G}_a &\to \mathrm{GL}_3(K) \\ x &\mapsto \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix}. \end{aligned}$$

Then

$$tu_{\alpha_1-\alpha_2}(x)t^{-1} = \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1^{-1} & & \\ & t_2^{-1} & \\ & & t_3^{-1} \end{pmatrix} = \begin{pmatrix} 1 & t_1 t_2^{-1} x & \\ & 1 & \\ & & 1 \end{pmatrix} = u_{\alpha_1-\alpha_2}((\alpha_1 - \alpha_2)(t)x),$$

as desired, so

$$U_{\alpha_1-\alpha_2} = u_{\alpha_1-\alpha_2}(\mathbb{G}_a) = \begin{pmatrix} 1 & * & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Table 1 shows the roots and the corresponding root spaces and one-dimensional unipotent subgroups. Each subalgebra (resp. subgroup) consists of all elements in $\mathfrak{gl}_3(K)$ (resp. $\mathrm{GL}_3(K)$) of the given form.

α	$\mathfrak{g}_\alpha$	U_α	$-\alpha$	$\mathfrak{g}_{-\alpha}$	$U_{-\alpha}$
$\alpha_1 - \alpha_2$	$\begin{pmatrix} 0 & x & \\ & 0 & \\ & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\alpha_2 - \alpha_1$	$\begin{pmatrix} 0 & & \\ x & 0 & \\ & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix}$
$\alpha_2 - \alpha_3$	$\begin{pmatrix} 0 & & \\ & 0 & x \\ & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & x \\ & & 1 \end{pmatrix}$	$\alpha_3 - \alpha_2$	$\begin{pmatrix} 0 & & \\ & 0 & \\ & x & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & x & 1 \end{pmatrix}$
$\alpha_1 - \alpha_3$	$\begin{pmatrix} 0 & & x \\ & 0 & \\ & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & x \\ & 1 & \\ & & 1 \end{pmatrix}$	$\alpha_3 - \alpha_1$	$\begin{pmatrix} 0 & & \\ & 0 & \\ x & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & \\ x & & 1 \end{pmatrix}$

Table 1: Roots, root spaces, and one-dimensional unipotent subgroups of $\mathrm{GL}_3(K)$

4.1.2 Weyl Group

A *monomial matrix* is a matrix with exactly one nonzero entry in each row and column.

Proposition 4.1. *$N_G(T)$ is the group of monomial matrices.*

Proof. Suppose $a = (a_{ij})$ normalizes T. Write m_{ij} for the minor determinants of a, so $\mathrm{adj}(a) = ((\pm 1)^{i+j} m_{ji})$. Since $\mathrm{adj}(a)$ is a scalar multiple of a^{-1}, we have $at\,\mathrm{adj}(a) \in T$ for any $t \in T$. Then

$$(at\,\mathrm{adj}(a))_{ij} = \sum_k (at)_{ik} (\mathrm{adj}(a))_{kj} = \sum_k t_k a_{ik} (\pm 1)^{k+j} m_{jk}. \tag{3}$$

If $i \neq j$, then this must vanish for all values of t_1, t_2, t_3, so $a_{ik} m_{jk} = 0$ for each $k = 1, 2, 3$.

Assume a has two nonzero values in the same row or column, say $a_{11}, a_{21} \neq 0$ after relabeling and transposing if necessary. Then $a_{11} m_{j1} = 0$ for $j \neq 1$ and $a_{21} m_{j1} = 0$ for $j \neq 2$, so $m_{j1} = 0$ for all $j = 1, 2, 3$. Thus $\mathrm{adj}(a)$ is not invertible, a contradiction. Since a is invertible, this forces a to be a monomial matrix.

Conversely, let a be a monimial matrix. If $a_{ik} \neq 0$, then for any $j \neq i$, the (i, j)-th minor contains a whole row of 0's (since $a_{il} = 0$ for all $l \neq k$), so $m_{ij} = 0$. Thus by (3), a normalizes T. $\square$

We already checked that $C_G(T) = T$. Every monomial matrix can be scaled with an appropriate diagonal matrix to a unique permutation matrix (monomial matrix with nonzero entries 1), so the Weyl group $W = N_G(T)/C_G(T)$ is naturally identified with the group of permutation matrices. The natural action on the standard basis, identified with $\{1, 2, 3\}$, yields the isomorphism $W \cong S_3$. Concretely,

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \longleftrightarrow \mathrm{id}, \qquad \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \longleftrightarrow (1\ 2), \qquad \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \longleftrightarrow (1\ 3),$$
$$\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \longleftrightarrow (2\ 3), \quad \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \longleftrightarrow (1\ 2\ 3), \quad \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} \longleftrightarrow (1\ 3\ 2).$$

4.1.3 Action of the Weyl Group

Let $i \neq j$. The singular tori $T_\alpha = (\ker \alpha)^\circ$ of codimension 1 in G are

$$T_{\alpha_i - \alpha_j} = T_{\alpha_j - \alpha_i} = \{t \in T : t_i = t_j\}.$$

We showed earlier that $C_G(T) = T$. By an analogous argument,

$$G_{\alpha_i-\alpha_j} := C_G(T_{\alpha_i-\alpha_j}) = \{(a_{kl}) \in \mathrm{GL}_3(K) : a_{kl} = 0 \text{ if } k \neq l \text{ and } (k,l) \neq (i,j), (j,i)\}.$$

We will check that the example

$$G_{\alpha_1-\alpha_2} = G_{\alpha_2-\alpha_1} = \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}$$

has the structure claimed in the general theory; calculations for other G_α will be similar. This is isomorphic to $\mathrm{GL}_2(K) \times \mathrm{GL}_1(K)$. By an argument similar to the one used for G, $R(G_{\alpha_1-\alpha_2})$ is the subgroup of diagonal matrices t with $t_1 = t_2$. This is $S \times \mathrm{GL}_1(K)$ in the above isomorphism, where S is the subgroup of scalar matrices in $\mathrm{GL}_2(K)$, so $G_{\alpha_1-\alpha_2}/R(G_{\alpha_1-\alpha_2}) \cong \mathrm{PGL}_2(K)$. Thus $G_{\alpha_1-\alpha_2}$ is reductive of semisimple rank 1. By the same calculation as for $\mathrm{GL}_3(K)$, $G_{\alpha_1-\alpha_2}$ has roots $\pm(\alpha_1 - \alpha_2)$. Since

$$N_{G_{\alpha_1-\alpha_2}}(T) = N_G(T) \cap G_{\alpha_1-\alpha_2} = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \cup \begin{pmatrix} & * & \\ * & & \\ & & * \end{pmatrix},$$

$G_{\alpha_1-\alpha_2}$ has Weyl group $\{\mathrm{id}, \sigma_{\alpha_1-\alpha_2}\}$ of order 2 represented by $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ and $n_{\alpha_1-\alpha_2} = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$. There are two standard Borels,

$$B_{\alpha_1-\alpha_2} = \begin{pmatrix} * & * & \\ & * & \\ & & * \end{pmatrix}, \quad B'_{\alpha_1-\alpha_2} = \begin{pmatrix} * & & \\ * & * & \\ & & * \end{pmatrix},$$

intersecting in T. Note that $U_{\alpha_1-\alpha_2}$ is the unipotent part of $B_{\alpha_1-\alpha_2}$. The unipotent part of each Borel subgroup is isomorphic to $\mathbb{G}_a$; for example, we have the isomorphism $\mathbb{G}_a \to B_{\alpha_1-\alpha_2}$ given by $x \mapsto \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix}$. We have

$$n_{\alpha_1-\alpha_2}^{-1} t n_{\alpha_1-\alpha_2} = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} = \begin{pmatrix} t_2 & & \\ & t_1 & \\ & & t_3 \end{pmatrix}.$$

Composing with this conjugation, $\sigma_{\alpha_1-\alpha_2}$ acts on X^* by fixing α_3 and exchanging α_1 and α_2, so the action on $\mathbb{R} \otimes X^*$ is given by $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 \mapsto c_2\alpha_1 + c_1\alpha_2 + c_3\alpha_3$. This sends $\alpha_1 - \alpha_2 \mapsto \alpha_2 - \alpha_1$ and fixes pointwise the codimension-one subspace $\mathbb{R}(\alpha_1 + \alpha_2) \oplus \mathbb{R}\alpha_3$, so $\sigma_{\alpha_1-\alpha_2}$ acts as a reflection relative to $\alpha_1 - \alpha_2$. Similarly, $\sigma_{\alpha_1-\alpha_2}$ acts on $\mathbb{R} \otimes X_*$ by $c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 \mapsto c_2\lambda_1 + c_1\lambda_2 + c_3\lambda_3$.

Since Φ is irreducible (of type A_2), there exists a unique (up to scalar multiplication) admissible inner product on $\mathbb{R} \otimes X^*$ and on $\mathbb{R} \otimes X_*$. Figure 1 shows Φ under this inner product. Note that $\sigma_{\alpha_1-\alpha_2}$ is then an orthogonal reflection.

For $x = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$,

$$x - \langle x, \lambda_1 - \lambda_2 \rangle (\alpha_1 - \alpha_2) = x - (c_1 - c_2)(\alpha_1 - \alpha_2) = c_2\alpha_1 + c_1\alpha_2 + c_3\alpha_3 = \sigma_{\alpha_1-\alpha_2}(x),$$

so $(\alpha_1 - \alpha_2)^\vee = \lambda_1 - \lambda_2$. Other coroots are similar, so $\Phi^\vee = \{\lambda_i - \lambda_j : i \neq j\}$. The span of $\Phi^\vee$ in $\mathbb{R} \otimes X_*$ is $\{c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 : c_1 + c_2 + c_3 = 0\}$.

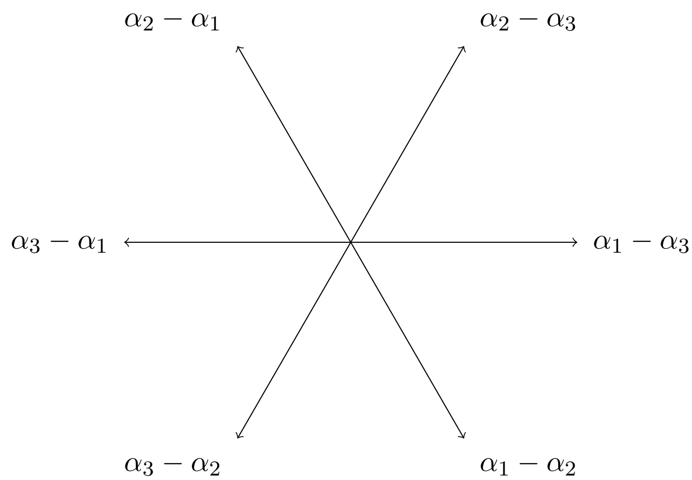

Figure 1: Root system of $\mathrm{GL}_3(K)$

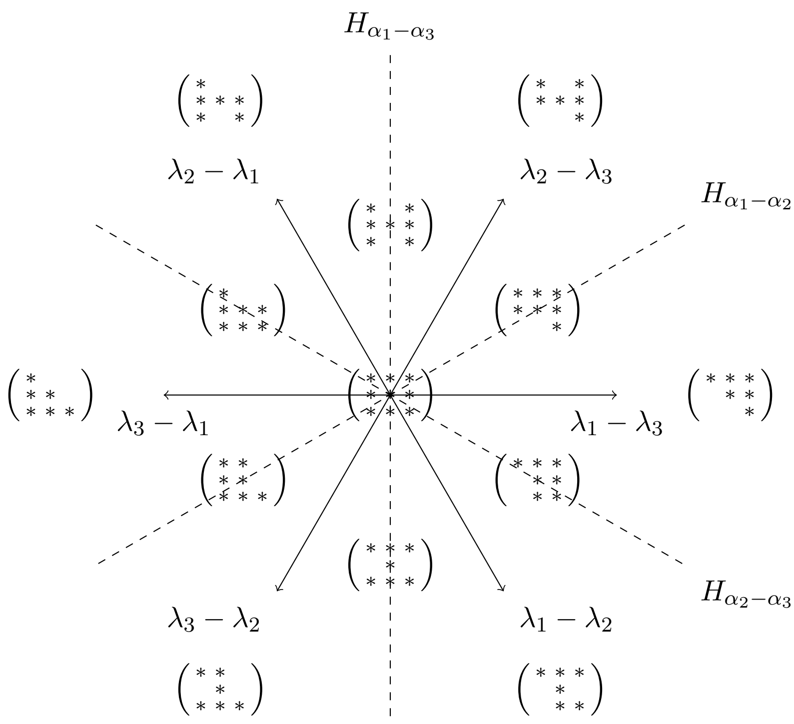

Figure 2: Coroots, spherical apartment, and associated parabolic subgroups of $\mathrm{GL}_3(K)$

We now describe the spherical apartment. Since $\langle \alpha_i - \alpha_j, c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 \rangle = c_i - c_j$, the hyperplanes are $H_{\alpha_i-\alpha_j} = H_{\alpha_j-\alpha_i} = \{\sum c_i\lambda_i : c_i = c_j\}$ and the regular cocharacters are $\sum c_i\lambda_i$ with c_i all distinct. Figure 2 shows the (reduced) spherical apartment with the coroots overlaid and the associated parabolic subgroup shown for each facet. The three hyperplanes (dotted) partition the regular cocharacters into six Weyl chambers. For example, let F be the Weyl chamber containing $\lambda_1 - \lambda_3$. Then F is the set of characters λ satisfying

$$\langle \lambda, \alpha_1 - \alpha_2 \rangle, \langle \lambda, \alpha_2 - \alpha_3 \rangle > 0.$$

Note that these automatically imply $\langle \lambda, \alpha_1 - \alpha_3 \rangle > 0$. This is a restatement of the simple geometric fact that, in the figure, a cocharacter lying below $H_{\alpha_1-\alpha_2}$ and above $H_{\alpha_2-\alpha_3}$ automatically lies to the right of $H_{\alpha_1-\alpha_3}$. In addition to the Weyl chambers, there are six facets of dimension one (in the reduced spherical apartment), given by the halves (separated by the origin) of each hyperplane, and one facet of dimension zero, the origin. The associated parabolic subgroups are easily calculated using the U_α determined earlier; for example,

$$P_F = \langle T, U_{\alpha_1-\alpha_2}, U_{\alpha_2-\alpha_3}, U_{\alpha_1-\alpha_3} \rangle = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}.$$

As we noted earlier, this is a Borel subgroup, as it should be since F is a maximal facet.

The action of W by conjugation on $\mathfrak{B}^T$ and on the set of parabolic subgroups containing T can also be seen through spherical apartment. For example, $\sigma_{\alpha_1-\alpha_3}$ acts on $\mathbb{R} \otimes X_*$ as an orthogonal reflection relative to $\lambda_1 - \lambda_3$, i.e. a reflection across $H_{\alpha_1-\alpha_3}$. In particular, it takes the Weyl chamber F, containing $\lambda_1 - \lambda_3$, to its opposite Weyl chamber, containing $\lambda_3 - \lambda_1$. Correspondingly, conjugation by the representative permutation matrix of $\sigma_{\lambda_1-\lambda_3}$ takes P_F to the opposite Borel subgroup:

$$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}^{-1} = \begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix}.$$

It is easily checked from such explicit description that W acts simply transitively on $\mathfrak{B}^T$.

4.1.4 Bruhat Decomposition

At least for $\mathrm{GL}_3(K)$, the Bruhat decomposition can be checked explicitly without too much effort. Let B be the Borel subgroup of upper triangular matrices. Clearly $BIB = B$. For non-trivial double cosets, for example, any element of $B \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} B$ has the form

$$\begin{pmatrix} \# & * & * \\ & \# & * \\ & & \# \end{pmatrix} \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} \# & * & * \\ & \# & * \\ & & \# \end{pmatrix} = \begin{pmatrix} \# & * & * \\ & \# & * \\ & & \# \end{pmatrix} \begin{pmatrix} & \# & * \\ \# & * & * \\ & & \# \end{pmatrix} = \begin{pmatrix} * & * & * \\ \# & * & * \\ & & \# \end{pmatrix},$$

where $\#$ indicates a nonzero element, and a computation shows that all matrices in $\mathrm{GL}_3(K)$ of this form appear in this double coset. Similarly, we find

$$B \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} B = \begin{pmatrix} \# & * & * \\ & \# & * \\ & & \# \end{pmatrix} \begin{pmatrix} \# & * & * \\ & & \# \\ & \# & * \end{pmatrix} = \begin{pmatrix} \# & * & * \\ & * & * \\ & \# & * \end{pmatrix}$$

$$B \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} B = \begin{pmatrix} \# & * & * \\ & \# & * \\ & & \# \end{pmatrix} \begin{pmatrix} & & \# \\ \# & * & * \\ & \# & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ \# & * & * \\ & \# & * \end{pmatrix}.$$

For $B\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix}B$ and $B\begin{pmatrix} & 1 & \\ & & 1\\ 1 & & \end{pmatrix}B$, consider elements

$$\begin{aligned}
\begin{pmatrix} \# & * & * \\ & \# & a \\ & & b \end{pmatrix}\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} c & d & * \\ & \# & * \\ & & \# \end{pmatrix} &= \begin{pmatrix} \# & * & * \\ & \# & a \\ & & b \end{pmatrix}\begin{pmatrix} & & \# \\ & \# & * \\ c & d & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ ac & \#+ad & * \\ bc & bd & * \end{pmatrix}\\
\begin{pmatrix} \# & * & * \\ & \# & a \\ & & b \end{pmatrix}\begin{pmatrix} & 1 & \\ & & 1\\ 1 & & \end{pmatrix}\begin{pmatrix} c & d & * \\ & \# & * \\ & & \# \end{pmatrix} &= \begin{pmatrix} \# & * & * \\ & \# & a \\ & & b \end{pmatrix}\begin{pmatrix} & \# & * \\ & & \# \\ c & d & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ ac & ad & * \\ bc & bd & * \end{pmatrix},
\end{aligned}$$

which shows that $(a_{ij}) \in B\begin{pmatrix} & 1 & \\ & & 1\\ 1 & & \end{pmatrix}B$ satisfy $a_{21}a_{32} = a_{22}a_{31}$ while we have inequality for $(a_{ij}) \in B\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix}B$. A more detailed computation shows that all matrices in $\mathrm{Sp}_4(K)$ of the form $\begin{pmatrix} * & * & * \\ * & * & * \\ \# & * & * \end{pmatrix}$ with these properties lie in these double cosets, so the Bruhat decomposition holds.

4.2 $\mathrm{SL}_3(K)$

The *special linear group* $\mathrm{SL}_n(K)$ is the subgroup of $\mathrm{GL}_n(K)$ consisting of matrices of determinant 1. It can be shown that differentiating the determinant gives the trace, so the corresponding Lie algebra $\mathfrak{sl}_n(K)$ consists of the *traceless matrices* (matrices of trace zero).

Clearly $(\mathrm{GL}_n(K), \mathrm{GL}_n(K)) \subset \mathrm{SL}_n(K)$. It can be shown that the other inclusion holds if $n \neq 2$ or $|K| \neq 2$ (which holds here since K is algebraically closed), so $\mathrm{SL}_n(K)$ is the derived group of $\mathrm{GL}_n(K)$; see for example [14]. Moreover, $\mathrm{SL}_n(K)$ is semisimple; arguing as in $\mathrm{GL}_n(K)$, the radical lies in the identity component of the subgroup of scalar matrices, which is just the identity matrix. We check the relation between the structures of the two groups against the general theory in Section 3.5.

Let $G = \mathrm{GL}_n(K)$ and $G' = \mathrm{SL}_n(K)$. Retain the notations $T, \Phi, \alpha_i, \lambda_i$, etc. for G. Primed symbols will denote the analogues for G'. The radical $R(G)$, the subgroup of scalar matrices, equals $Z(G)^\circ (= Z(G))$ and is a central torus. Then $G = G'R(G)$ with finite intersection $R(G) \cap G' = \{tI : t^n = 1\}$. Since $\ker(\alpha_i - \alpha_j) = \{t \in T : t_i = t_j\}$, we also have $Z(G) = \bigcap_{\alpha\in\Phi} \ker \alpha$.

To simplify notation, we now assume $n = 3$. The diagonal matrices again form a maximal torus

$$T' = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} = T \cap G'.$$

This is the subtorus of T generated by the images of coroots $\lambda_i - \lambda_j$, $i \neq j$, of G. Composition with the inclusion $T' \hookrightarrow T$ defines a $\mathbb{Z}$-module homomorphism $\varphi : X^*(T) \to X^*(T')$. Let $\alpha_i' : T' \to \mathbb{G}_m$ be image of α_i, so $\alpha_i'(t) = t_i$; these generate $X^*(T')$, so φ is surjective. Since $t \in T'$ if and only if $t_1t_2t_3 = 1$, φ has kernel $\mathbb{Z}(\alpha_1 + \alpha_2 + \alpha_3)$ and induces a $\mathbb{Z}$-module isomorphism

$$X^*(T') \cong (\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3)/(\mathbb{Z}(\alpha_1 + \alpha_2 + \alpha_3)).$$

Recall that $\Phi = \{\alpha_i - \alpha_j : i \neq j\}$ and $\Phi^\vee = \{\lambda_i - \lambda_j : i \neq j\}$. These span, respectively,

$$\begin{aligned}
Q &= \{c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 : c_1 + c_2 + c_3 = 0\}\\
Q^\vee &= \{c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 : c_1 + c_2 + c_3 = 0\},
\end{aligned}$$

which are rank-two submodules of the rank-three $\mathbb{Z}$-modules $X^*(T)$ and $X_*(T)$, respectively. Then $(Q^\vee)^\perp = \mathbb{Z}(\alpha_1+\alpha_2+\alpha_3)$, so $X^*(T') \cong X^*/(X^\vee)^\perp$. The calculation of roots is unchanged from G, so $\Phi' = \{\alpha'_i - \alpha'_j : i \neq j\}$ with one-dimensional rootspaces $\mathfrak{g}_{\alpha'_i-\alpha'_j} = \langle E_{ij}\rangle$; see Table 1 and Figure 1, with the implied intersection now with G' instead of G. Moreover, in $\mathbb{R}\otimes X^*$ with the unique (up to scalar) admissible inner product, $\mathbb{Z}(\alpha_1+\alpha_2+\alpha_3)$ is the orthogonal complement of Q. Thus $X^*(T')$ is obtained from $X^*(T)$ by quotienting by the orthogonal complement of Q. In particular, this maps Φ injectively into $X^*(T')$, identifying it with Φ'. Moreover, the image Q' of Q now has full rank in the rank-two $\mathbb{Z}$-module $X^*(T')$.

Meanwhile, $X_*(T')$ is the submodule of $X_*(T)$ of cocharacters with image in T', so

$$X_*(T') = \{c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 : c_1+c_2+c_3 = 0\},$$

which equals $\widetilde{Q^\vee}$. The dual pairing is induced from that of G, i.e. $\langle \alpha'_i, \lambda_j\rangle = \delta_{ij}$ extended linearly; this is well defined because in fact only elements of $X_*(T')$ appear in the second entry. Again as in G, we calculate $(\Phi^\vee)' = \{\lambda_i - \lambda_j : i \neq j\}$. As with the root lattice, while $\operatorname{rk} Q^\vee < \operatorname{rk} X_*(T)$ (rank as $\mathbb{Z}$-modules), we have $\operatorname{rk}(Q^\vee)' = \operatorname{rk} X_*(T')$ for the span $(Q^\vee)'$ of $(\Phi^\vee)'$. Thus going to the derived group preserves the root and coroot systems while quotienting out the complements of their spans. In particular, the (reduced) spherical apartment of G' can be identified with that of G, and the associated parabolic subgroups also follow the same calculation; see Figure 2.

4.3 $\mathrm{PSL}_3(K)$

For K algebraically closed, we may view $\mathrm{PSL}_3(K)$ as $\mathrm{SL}_3(K)/S$, where $S = \{aI : a^3 = 1\}$ is the subgroup of scalar matrices. Since S is finite, the canonical map $\pi : \mathrm{SL}_3(K) \to \mathrm{PSL}_3(K)$ is an isogeny. In fact, $\mathrm{SL}_3(K)$ and $\mathrm{PSL}_3(K)$ are the only semisimple groups of type A_2, being respectively simply-connected and adjoint. These serve as examples of the discussion of isogeny in Section 3.4.

Retain the notations of the previous section, with primed symbols to denote objects defined for $G' = \mathrm{SL}_3(K)$. Recall that

$$\begin{aligned} X^*(T') &= (\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3)/(\mathbb{Z}(\alpha_1+\alpha_2+\alpha_3)) \\ X_*(T') &= \{c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 : c_i \in \mathbb{Z}, c_1+c_2+c_3 = 0\}. \end{aligned}$$

In the basis $X^*(T') = \mathbb{Z}\alpha'_1 \oplus \mathbb{Z}\alpha'_2$, where α'_i is the image of α_i, we compute

$$\begin{aligned} \Phi' &= \{\alpha'_i - \alpha'_j : i \neq j\} = \{\pm(\alpha'_1 - \alpha'_2), \pm(2\alpha'_1 + \alpha'_2), \pm(\alpha'_1 + 2\alpha'_2)\} \\ Q' &= \{c_1\alpha'_1 + c_2\alpha'_2 : c_i \in \mathbb{Z},\ c_1 + c_2 \equiv 0 \pmod 3\}. \end{aligned}$$

An element $c_1\alpha'_1 + c_2\alpha'_2 \in \mathbb{R}\otimes X^*(T')$, $c_i \in \mathbb{R}$, lies in the weight lattice P' if and only if $\langle c_1\alpha'_1 + c_2\alpha'_2, \lambda_i - \lambda_j\rangle = c_i - c_j \in \mathbb{Z}$ for all $i \neq j$ (and where $c_3 = 0$), so $P' = X^*(T')$. Thus G' is simply-connected. If we are careful to keep in mind that $\alpha'_1 + \alpha'_2 + \alpha'_3 = 0$, we may carry out these computations more symmetrically:

$$\begin{aligned} Q' &= \{c_1\alpha'_1 + c_2\alpha'_2 + c_3\alpha'_3 : c_i \in \mathbb{Z},\ c_1+c_2+c_3 \equiv 0 \pmod 3\} \\ X^*(T') = P' &= \{c_1\alpha'_1 + c_2\alpha'_2 + c_3\alpha'_3 : c_i \in \mathbb{Z}\}. \end{aligned}$$

Meanwhile, $\pi(G') = \mathrm{PSL}_3(K)$ has maximal torus $\pi(T') = T'/S$. Composition with the surjection $\pi : T' \to \pi(T')$ defines an injective $\mathbb{Z}$-module homomorphism $\psi : X^*(\pi(T')) \to X^*(T')$. The image consists of characters of T' that factor through π, i.e. kill S. This means

$$(c_1\alpha_1' + c_2\alpha_2' + c_3\alpha_3')(aI) = a^{c_1+c_2+c_3} = 1$$

for any third root of unity a, so the image of ψ is exactly Q'. Since Q' has full rank in $X^*(T')$, we may identify $\mathbb{R} \otimes X^*(\pi(T'))$ with $\mathbb{R} \otimes X^*(T')$ via ψ. Thus

$$\mathbb{R} \otimes X^*(\pi(T')) \cong (\mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2 \oplus \mathbb{R}\alpha_3)/\mathbb{R}(\alpha_1 + \alpha_2 + \alpha_3),$$

and writing α_i'' for the image of α_i, we have

$$X^*(\pi(T')) = \{c_1\alpha_1'' + c_2\alpha_2'' + c_3\alpha_3'' : c_i \in \mathbb{Z},\ c_1 + c_2 + c_3 \equiv 0 \pmod 3\},$$

where $c_1\alpha_1'' + c_2\alpha_2'' + c_3\alpha_3''$, when it lies in $X^*(\pi(T'))$, is the preimage of $c_1\alpha_1' + c_2\alpha_2' + c_3\alpha_3'$ under ψ.

The adjoint action (i.e. conjugation) of T' on G' induces that of $\pi(T')$ on $\pi(G')$. In exact analogy with G', we compute

$$\begin{aligned}
\Phi'' &= \{\alpha_i'' - \alpha_j'' : i \neq j\} \\
X^*(\pi(T')) = Q'' &= \{c_1\alpha_1'' + c_2\alpha_2'' + c_3\alpha_3'' : c_i \in \mathbb{Z},\ c_1 + c_2 + c_3 \equiv 0 \pmod 3\} \\
P'' &= \{c_1\alpha_1'' + c_2\alpha_2'' + c_3\alpha_3'' : c_i \in \mathbb{Z}\}.
\end{aligned}$$

In particular, $\pi(G')$ is adjoint.

Note that the map

$$\begin{aligned}
P' &\to \mathbb{Z}/3\mathbb{Z} \\
c_1\alpha_1' + c_2\alpha_2' + c_3\alpha_3' &\mapsto (c_1 + c_2 + c_3) + 3\mathbb{Z}
\end{aligned}$$

and the analogue for P'' induce isomorphisms $P'/Q' \cong \mathbb{Z}/3\mathbb{Z} \cong P''/Q''$. Since $\mathbb{Z}/3\mathbb{Z}$ has no nontrivial subgroup (equivalently, no proper intermediate lattice exists between P' and Q' and between P'' and Q''), G' and $\pi(G')$ account for all possible fundamental groups, hence all semisimple groups of type A_2. More generally, the same calculations show that A_{n-1} has fundamental group $\mathbb{Z}/n\mathbb{Z}$, and that $\mathrm{SL}_n(K)$ and $\mathrm{PSL}_n(K)$ are the simply-connected and adjoint semisimple groups of this type.

Section 4.5 below looks at $\mathrm{Sp}_4(K)$. In analogy with the current section, it can be shown that $\mathrm{Sp}_4(K)$ and $\mathrm{PSp}_4(K)$ are the simply-connected and adjoint semisimple groups of type C_2.

4.4 Interlude on Bilinear Forms

A *bilinear form* on a K-vector space V is a map $B : V \times V \to K$ that is K-linear in each entry. It is *symmetric* if $B(v, w) = B(w, v)$ for all $v, w \in V$ and *alternating* if $B(v, v) = 0$ for all $v \in V$. In characteristic $\neq 2$, the last condition is equivalent to $B(v, w) = -B(w, v)$ for all $v, w \in V$.

Let $\dim V = n$, and fix a bilinear form B on V. Given a fixed basis of V, we can write $B(v,w) = x^t M y$ for a unique matrix M, where x (resp. y) is the coordinate vector of v (resp. w) under this basis, viewed as a $1 \times n$ matrix; we say that B is *given by* M (under this basis). Given another basis with transition matrix P, i.e. so that Px (resp. Py) is the new coordinate vector of v (resp. w), we have $B(v,w) = (Px)^t M(Py) = x^t(P^t MP)y$, so B is now given by $P^t MP$. In calculations, we will choose a convenient basis to simplify the corresponding matrix.

The *isometry group* or *stabilizer group* of B is the group of automorphisms φ of V that preserve B, that is, $B(\varphi v, \varphi w) = B(v,w)$. Fix a basis of V. If B is given by M, these are invertible linear transformations given by matrices A satisfying $A^t MA = M$, or equivalently the fixed points of the involution $A \mapsto (MA^{-1}M^{-1})^t$ of $\mathrm{GL}_n(K)$. Differentiating $A^t MA = M$ shows that the corresponding Lie algebra, identified with a subalgebra of $\mathfrak{gl}_n(K)$, consists of matrices A satisfying $A^t M + MA = 0$.

Two bilinear forms B, B' on V are called *equivalent* if they differ by a change of basis, i.e. if there exists some automorphism f of V such that $B(fv, fw) = B'(v,w)$ for all $v, w \in V$. Then the isometry groups of B and B' are isomorphic via conjugation by f; indeed, φ in the isometry group of B' implies $B(f\varphi f^{-1}v, f\varphi f^{-1}w) = B'(\varphi f^{-1}v, \varphi f^{-1}w) = B'(f^{-1}v, f^{-1}w) = B(v,w)$, and similarly for the inverse map. It is also useful to rephrase equivalence using coordinates. Fix a basis of V. Writing $B(v,w) = x^t My$ as before and P for the matrix representation of f, we get $B'(v,w) = B(fv, fw) = (Px)^t M(Py) = x^t(P^t MP)y$. Note that this relation between matrices giving equivalent bilinear forms is the same as that between the matrices of one bilinear form under two bases.

The *kernel* of B is the subspace of V of vectors v with $B(v,w) = 0$ for all $w \in V$. A bilinear form is called *nondegenerate* if its kernel is trivial, and *degenerate* otherwise. We will only be concerned with nondegenerate bilinear forms.

4.5 $\mathrm{Sp}_4(K)$

The *symplectic group* $\mathrm{Sp}_m(K)$, m even, is the isometry group of a nondegenerate alternating bilinear form on K^m. For m fixed, we may speak of *the* symplectic group for the following reason: for arbitrary K (not necessarily algebraically closed), it can be shown that no such form exists for m odd, and that for $m = 2n$ even, there exists a unique such form up, defining $\mathrm{Sp}_{2n}(K)$ uniquely in $\mathrm{GL}_{2n}(K)$, up to change of basis. Moreover, it can be shown that elements of $\mathrm{Sp}_{2n}(K)$ have determinant one, and that $\mathrm{Sp}_{2n}(K)$ is connected. Here we look at $\mathrm{Sp}_4(K)$ for the bilinear form given by[10]

$$M = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

[10]This choice of M and the obvious higher-rank analogues preserve the standard maximal torus and Borel subgroup of $\mathrm{GL}_m(K)$ (upper triangular matrices). Compare for example $\mathrm{antidiag}(I, -I)$, which preserves only the maximal torus. Recall that a choice of Borel subgroup determines a base of the root system. Our choice of M also preserves the *standard pinning* of $\mathrm{GL}_m(K)$, a certain explicit basis for the rootspaces of simple roots (here the superdiagonal elements $E_{i,i+1}$). These facts aid in some calculations.

Higher-rank analogues will be similar. Where convenient, we will write $G = \mathrm{Sp}_4(K)$ and $\mathfrak{g} = \mathfrak{sp}_4(K)$.

For a diagonal matrix t, the condition to lie in $\mathrm{Sp}_4(K)$ is $tMt = M$. Since

$$tMt = \begin{pmatrix} & & & t_1t_4 \\ & & -t_2t_3 & \\ & t_3t_2 & & \\ -t_4t_1 & & & \end{pmatrix},$$

this happens exactly when $t_1t_4 = t_2t_3 = 1$. Hence

$$T := \mathrm{Sp}_4(K) \cap D = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix} : t_1, t_2 \in K^* \right\}.$$

This is in fact a maximal torus, so $\mathrm{Sp}_4(K)$ has rank 2. For a diagonal matrix t, we will write $t = \mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1})$. Define $\alpha_1, \alpha_2 : T \to \mathbb{G}_m$ by $\alpha_i(t) = t_i$ and $\lambda_i : \mathbb{G}_m \to T$ by

$$\lambda_1(x) = \begin{pmatrix} x & & & \\ & 1 & & \\ & & 1 & \\ & & & x^{-1} \end{pmatrix}, \quad \lambda_2(x) = \begin{pmatrix} 1 & & & \\ & x & & \\ & & x^{-1} & \\ & & & 1 \end{pmatrix}.$$

Then $X^*(T) = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ and $X_*(T) = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ with dual pairing $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$.

Computing $A^tM + MA = 0$ shows that the Lie algebra $\mathfrak{sp}_4(K)$ consists of matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & -a_{13} \\ a_{31} & a_{32} & -a_{22} & a_{12} \\ a_{41} & -a_{31} & a_{21} & -a_{11} \end{pmatrix}.$$

From this, we can easily calculate the roots as before:

$$\Phi = \{\pm 2\alpha_1, \pm 2\alpha_2, \pm(\alpha_1 - \alpha_2), \pm(\alpha_1 + \alpha_2)\} = \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq 2\},$$

allowing $i = j$. In fact, the last expression generalizes to $\mathrm{Sp}_{2n}(K)$ with $1 \leq i, j \leq n$, and Φ is then an irreducible root system of type C_n. Focusing on $n = 2$, we carry out computations for one each of the two different types of roots: $2\alpha_1$ (when $i = j$) and $\alpha_1 + \alpha_2$ (when $i \neq j$).

Since

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix} \begin{pmatrix} 0 & & & x \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & & & t_1^2x \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = (2\alpha_1)(t) \begin{pmatrix} 0 & & & x \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

$2\alpha_1$ is a root. Since $(xE_{14})^2 = 0$, we have $\exp(xE_{14}) = 1 + xE_{14}$, so $U_{2\alpha_1}$ is the image of

$$\begin{aligned} u_{2\alpha_1} : \mathbb{G}_a &\to \mathrm{GL}_3(K) \\ x &\mapsto \begin{pmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \end{aligned}$$

satisfying as desired $tu_{2\alpha_1}(x)t^{-1} = u_{2\alpha_1}((2\alpha_1)(t)x)$. Moreover,

$$T_{2\alpha_1} := (\ker(2\alpha_1))^\circ = \left\{ \begin{pmatrix} 1 & & & \\ & x & & \\ & & x & \\ & & & 1 \end{pmatrix} : x \in K^* \right\}, \quad G_{2\alpha_1} := C_G(T_{2\alpha_1}) = \begin{pmatrix} * & & & * \\ & * & & \\ & & * & \\ * & & & * \end{pmatrix},$$

and $G_{2\alpha_1}$ is (by an analogous argument as in $\mathrm{GL}_3(K)$) reductive of semisimple rank 1 and has Borel subgroups

$$B_{2\alpha_1} = \begin{pmatrix} * & & & * \\ & * & & \\ & & * & \\ & & & * \end{pmatrix}, \quad B'_{2\alpha_1} = \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ * & & & * \end{pmatrix},$$

with $U_{2\alpha_1}$ the unipotent part of $B_{2\alpha_1}$.

For the type $i \neq j$, we have for example the root $\alpha_1 + \alpha_2$ since

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix} \begin{pmatrix} 0 & x & & \\ & 0 & & -x \\ & & 0 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & t_1t_2x & & \\ & 0 & & -t_1t_2x \\ & & 0 & \\ & & & 0 \end{pmatrix} = (\alpha_1+\alpha_2)(t) \begin{pmatrix} 0 & x & & \\ & 0 & & -x \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Note that, although we need both entries x and $-x$ to obtain an element of $\mathfrak{sp}_4(K)$, the rootspace remains one-dimensional. Exponentiating, squares and higher powers again vanishes, so $U_{\alpha_1+\alpha_2}$ is the image of

$$\begin{aligned} u_{\alpha_1+\alpha_2} : \mathbb{G}_a &\to \mathrm{Sp}_4(K) \\ x &\mapsto \begin{pmatrix} 1 & x & & \\ & 1 & & -x \\ & & 1 & \\ & & & 1 \end{pmatrix}, \end{aligned}$$

which satisfies $tu_{\alpha_1+\alpha_2}(x)t^{-1} = u_{\alpha_1+\alpha_2}((\alpha_1+\alpha_2)(t)x)$, and

$$T_{\alpha_1+\alpha_2} := \ker(\alpha_1+\alpha_2)^\circ = \left\{ \begin{pmatrix} x & & & \\ & x^{-1} & & \\ & & x & \\ & & & x^{-1} \end{pmatrix} : x \in K^* \right\}, \quad G_{\alpha_1+\alpha_2} := C_G(T_{\alpha_1+\alpha_2}) = \begin{pmatrix} * & * & & \\ * & * & & * \\ & & * & * \\ & & * & * \end{pmatrix},$$

$$B_{\alpha_1+\alpha_2} = \begin{pmatrix} * & * & & \\ & * & & * \\ & & * & \\ & & & * \end{pmatrix}, \quad B'_{\alpha_1+\alpha_2} = \begin{pmatrix} * & & & \\ * & * & & \\ & & * & \\ & * & & * \end{pmatrix}.$$

It is easily checked that $U_{\alpha_1+\alpha_2}$ contains all elements in $\mathrm{Sp}_4(K)$ of the form $\begin{pmatrix} 1 & * & & \\ & 1 & & * \\ & & 1 & \\ & & & 1 \end{pmatrix}$, hence equals the unipotent part of $B_{\alpha_1+\alpha_2}$.

The roots and corresponding root spaces and one-dimensional unipotent subgroups are shown in Table 2.

We now determine the Weyl group. Essentially the same argument as for $\mathrm{GL}_3(K)$ shows that $C_G(T) = T$. We show that $N_G(T)$ is again the subgroup of monomial matrices. Any monomial matrix normalizes T. Conversely, if (a_{ij}) normalizes T, then $\sum_k t_k a_{ik}(\pm 1)^{k+j} m_{jk} = 0$ for all $i \neq j$ and all values of t_k. Despite the restriction $t_1t_4 = t_2t_3 = 1$ for $\mathrm{Sp}_4(K)$, this still shows that $a_{ik}m_{jk} = 0$ for all $i \neq j$ and all k. Indeed, the equation becomes

$$-t_1 a_{j1} m_{k1} + t_2 a_{j2} m_{k2} - \frac{1}{t_2} a_{i3} m_{k3} + \frac{1}{t_1} a_{i4} m_{j4} = 0.$$

α	$\mathfrak{g}_\alpha$	U_α	$-\alpha$	$\mathfrak{g}_{-\alpha}$	$U_{-\alpha}$
$2\alpha_1$	$\begin{pmatrix} 0 & & & x \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$-2\alpha_1$	$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ x & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x & & & 1 \end{pmatrix}$
$2\alpha_2$	$\begin{pmatrix} 0 & & & \\ & 0 & x & \\ & & 0 & \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$2\alpha_2$	$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & x & 0 & \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{pmatrix}$
$\alpha_1 - \alpha_2$	$\begin{pmatrix} 0 & x & & \\ & 0 & & \\ & & 0 & x \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & x \\ & & & 1 \end{pmatrix}$	$\alpha_2 - \alpha_1$	$\begin{pmatrix} 0 & & & \\ x & 0 & & \\ & & 0 & \\ & & x & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & \\ & & x & 1 \end{pmatrix}$
$\alpha_1 + \alpha_2$	$\begin{pmatrix} 0 & & x & \\ & 0 & & -x \\ & & 0 & \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & x & \\ & 1 & & -x \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$\alpha_1 + \alpha_2$	$\begin{pmatrix} 0 & & & \\ & 0 & & \\ x & & 0 & \\ & -x & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ x & & 1 & \\ & -x & & 1 \end{pmatrix}$

Table 2: Roots, root spaces, and one-dimensional unipotent subgroups of $\mathrm{Sp}_4(K)$

Fixing t_2 and varying t_1 shows that contribution from the two t_1 terms is constant. If $x, y \in K^*$ are two distinct values for t_1, then

$$\begin{aligned} -xa_{i1}m_{j1} + \frac{1}{x}a_{i4}m_{j4} &= -ya_{i1}m_{j1} + \frac{1}{yx}a_{i4}m_{j4} \\ \Rightarrow (y-x)a_{i1}m_{j1} &= \frac{x-y}{xy}a_{i4}m_{j4} \end{aligned}$$

and since xy can take on at least two values, we must have $a_{i1}m_{j1} = a_{i4}m_{j4} = 0$. The rest of the argument goes through as in $\mathrm{GL}_3(K)$.

By multiplying by a diagonal matrix in $\mathrm{Sp}_4(K)$, a monomial matrix can be given nonzero entries ± 1 in the first two rows, from which it follows that all nonzero entries are ± 1. It is then straightforward to compute which of these matrices lie in $\mathrm{Sp}_4(K)$ and are distinct modulo T; we find that W has eight elements represented by

$$n_{2\alpha_1} = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{pmatrix}, \quad n_{2\alpha_2} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix}, \quad n_{\alpha_1-\alpha_2} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & -1 \\ & & -1 & \end{pmatrix}, \quad n_{\alpha_1+\alpha_2} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

$$r_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad r_{\pi/2} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & & & -1 \\ & 1 & & \end{pmatrix}, \quad r_\pi = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}, \quad r_{3\pi/2} = \begin{pmatrix} & 1 & & \\ & & & 1 \\ 1 & & & \\ & & -1 & \end{pmatrix},$$

where σ_α, represented by n_α, acts on $\mathbb{R} \otimes X^*$ as a reflection relative to α. For example,

$$n_{2\alpha_1}^{-1} \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix} n_{2\alpha_1} = \begin{pmatrix} & & & -1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix} \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix} \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{pmatrix} = \begin{pmatrix} t_1^{-1} & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1 \end{pmatrix}$$

$$n_{\alpha_1+\alpha_2}^{-1} \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix} n_{\alpha_1+\alpha_2} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix} \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} = \begin{pmatrix} t_2^{-1} & & & \\ & t_1^{-1} & & \\ & & t_1 & \\ & & & t_2 \end{pmatrix},$$

so $\sigma_{2\alpha_1}$ acts by $\alpha_1 \mapsto -\alpha_1$, $\alpha_2 \mapsto \alpha_2$, and $\sigma_{\alpha_1+\alpha_2}$ acts by $\alpha_1 \mapsto -\alpha_2$, $\alpha_2 \mapsto -\alpha_1$. Figure 3 shows the root system with the unique (up to scalar) admissible inner product on $\mathbb{R} \otimes X^*$. Observe that

σ_α are the reflections claimed and that the other four elements r_θ act as rotations through angle θ, giving $W \cong D_8$, the dihedral group of order 8.

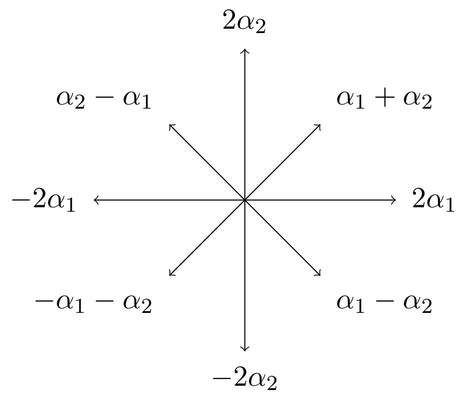

Figure 3: Root system of $\mathrm{Sp}_4(K)$

Figure 4 shows the coroots, spherical apartment, and associated parabolic subgroups under the unique (up to scalar) admissible inner product. As in $\mathrm{GL}_3(K)$, it is easy to see that the action of W by conjugation, on $\mathfrak{B}^T$ and on the set of parabolic subgroups containing T, corresponds via the spherical apartment to the action of W on $\mathbb{R} \otimes X_*$.

For $x = c_1\alpha_1 + c_2\alpha_2$,

$$\begin{aligned} x - \langle x, \lambda_1 \rangle (2\alpha_1) &= x - 2c_1\alpha_1 = -c_1\alpha_1 + c_2\alpha_2 = \sigma_{2\alpha_1}(x) \\ x - \langle x, \lambda_1 + \lambda_2 \rangle (\alpha_1 + \alpha_2) &= x - (c_1 + c_2)(\alpha_1 + \alpha_2) = -c_2\alpha_1 - c_1\alpha_2 = \sigma_{\alpha_1+\alpha_2}(x), \end{aligned}$$

so $(2\alpha_1)^\vee = \lambda_1$, $(\alpha_1 + \alpha_2)^\vee = \lambda_1 + \lambda_2$, and similarly for other roots of each type. Thus

$$\Phi^\vee = \{\pm\lambda_1, \pm\lambda_2, \pm\lambda_1 \pm \lambda_2\} = \{\pm\lambda_i, \pm\lambda_i \pm \lambda_j : 1 \le i, j \le 2, i \ne j\},$$

where the last expression holds for any $\mathrm{Sp}_{2n}(K)$ with $1 \le i, j \le n, i \ne j$. The root lattice Q is the span of $\Phi = \{\pm 2\alpha_1, \pm 2\alpha_2, \pm\alpha_1 \pm \alpha_2\}$. For $\alpha = c_1\alpha_1 + c_2\alpha_2 \in \mathbb{R} \otimes X^*$, $c_i \in \mathbb{R}$, we have $\alpha \in P$ if and only if $\langle \alpha, \pm\lambda_i \rangle = \pm c_i$ and $\langle \alpha, \pm\lambda_i \pm \lambda_j \rangle = \pm c_i \pm c_j$ are integral, so

$$\begin{aligned} Q &= \{c_1\alpha_1 + c_2\alpha_2 : c_i \in \mathbb{Z},\ c_1 + c_2 \equiv 0 \pmod 2\} \\ X^*(T) = P &= \{c_1\alpha_1 + c_2\alpha_2 : c_i \in \mathbb{Z}\}. \end{aligned}$$

Thus $\mathrm{Sp}_4(K)$ is simply-connected.

4.6 Orthogonal Groups

An *orthogonal group* is the isometry group of a symmetric bilinear form. Unlike with alternating bilinear forms, there will in general be multiple inequivalent nondegenerate symmetric bilinear

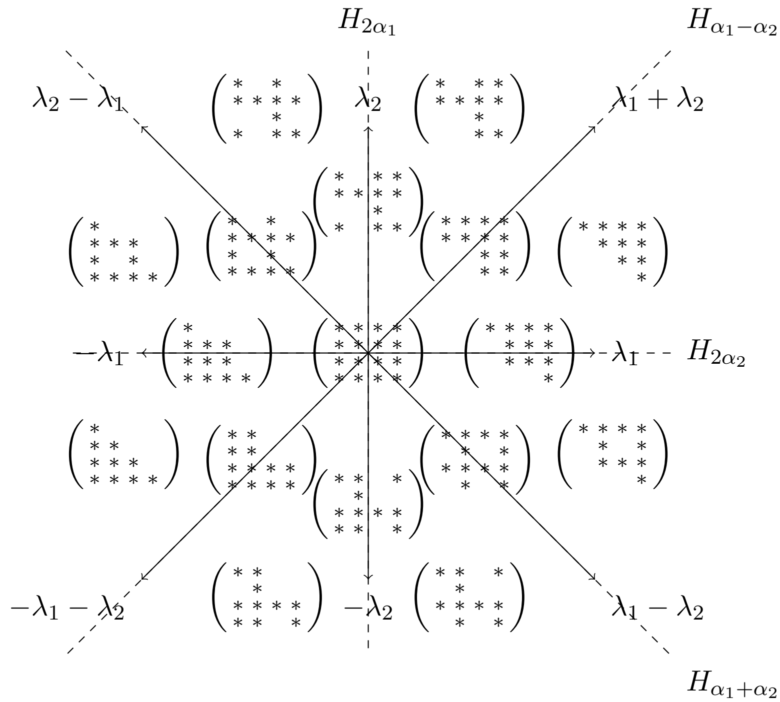

Figure 4: Coroots, spherical apartment, and associated parabolic subgroups of $\mathrm{Sp}_4(K)$

forms on a given vector space over a non-algebraically closed field. However, every such form has an orthogonal basis, making the associated matrix diagonal. Scaling a basis vector scales the corresponding diagonal matrix entry by a square, so that equivalences depend on which field elements are squares. We assume here that the characteristic of the field is not two; for a discussion of quadratic forms and orthogonal groups in characteristic two, see Sections 23.5-6 of [1].

For K algebraically closed, every element is a square. The orthogonal group for the therefore unique (up to change of basis) nondegenerate bilinear form on K^n is denoted $\mathrm{O}_n(K)$ (or sometimes $\mathrm{GO}_n(K)$ in analogy with $\mathrm{GL}_n(K)$). Unlike in $\mathrm{Sp}_{2n}(K)$, elements of $\mathrm{O}_n(K)$ have determinant ± 1 (distinct in characteristic $\neq 2$), comprising two connected components. The identity component, consisting of elements of determinant one, is called the *special orthogonal group* $\mathrm{SO}_n(K)$. Here we look at $\mathrm{SO}_5(K)$ for the bilinear form given by

$$M = \begin{pmatrix} &&&&1\\ &&&-1&\\ &&1&&\\ &-1&&&\\ 1&&&& \end{pmatrix}.$$

Higher-rank analogues for n odd will be similar.[11]

For a diagonal matrix t, the condition to lie in $\mathrm{O}_5(K)$ is $tMt = M$. Since

$$tMt = \begin{pmatrix} &&&&t_1t_5\\ &&&-t_2t_4&\\ &&t_3^2&&\\ &-t_4t_2&&&\\ t_5t_1&&&& \end{pmatrix},$$

this happens exactly when $t_1t_5 = t_2t_4 = 1$ and $t_3 = \pm 1$. Since $\det t = (t_1t_5)(t_2t_4)t_3$, we have $t_3 = 1$ for $t \in \mathrm{SO}_5(K)$. Hence

$$T := \mathrm{SO}_5(K) \cap D = \left\{ \begin{pmatrix} t_1&&&&\\ &t_2&&&\\ &&1&&\\ &&&t_2^{-1}&\\ &&&&t_1^{-1} \end{pmatrix} : t_1, t_2 \in K^* \right\}.$$

This is in fact a maximal torus, so $\mathrm{SO}_5(K)$ has rank 2. Note the similarity with $\mathrm{Sp}_4(K)$. Defining α_i, λ_j as in $\mathrm{Sp}_4(K)$, we have $X^*(T) = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ and $X_*(T) = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ with dual pairing $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$.

Since

$$A^tM + MA = \begin{pmatrix} a_{51} & -a_{41} & a_{31} & -a_{21} & a_{11}\\ a_{52} & -a_{42} & a_{32} & -a_{22} & a_{12}\\ a_{53} & -a_{43} & a_{33} & -a_{23} & a_{13}\\ a_{54} & -a_{44} & a_{34} & -a_{24} & a_{14}\\ a_{55} & -a_{45} & a_{35} & -a_{25} & a_{15} \end{pmatrix} + \begin{pmatrix} a_{51} & a_{52} & a_{53} & a_{54} & a_{55}\\ -a_{41} & -a_{42} & -a_{43} & -a_{44} & -a_{45}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35}\\ -a_{21} & -a_{22} & -a_{23} & -a_{24} & -a_{25}\\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix},$$

the condition $A^tM + MA = 0$ implies, using for the diagonal entries that the characteristic is not two, that the Lie algebra $\mathfrak{so}_5(K)$ consists of matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0\\ a_{21} & a_{22} & a_{23} & 0 & a_{14}\\ a_{31} & a_{32} & 0 & a_{23} & -a_{13}\\ a_{41} & 0 & a_{32} & -a_{22} & a_{12}\\ 0 & a_{41} & -a_{31} & a_{21} & -a_{11} \end{pmatrix}.$$

[11] As in $\mathrm{Sp}_4(K)$, this choice of M preserve the standard maximal torus, Borel subgroup, and pinning of $\mathrm{GL}_n(K)$, n odd. The analogues for n even are not symmetric, so we need another choice of M.

From this, one easily calculates the roots as before:

$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 - \alpha_2), \pm(\alpha_1 + \alpha_2)\} = \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq 2\},$$

allowing $i = j$. The last expression holds for any $\mathrm{SO}_{2n+1}(K)$ with $1 \leq i, j \leq n$, and Φ is then an irreducible root system of type B_n.

Root vectors for the roots $\pm(\alpha_1 - \alpha_2), \pm(\alpha_1 + \alpha_2)$ still vanish when squared; for example,

$$\mathfrak{g}_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & & & x & \\ & 0 & & & x \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}.$$

The remaining roots produce up to nonvanishing square terms, leading for the first time to nonlinear entries in the one-dimensional unipotent subgroups U_α. For example, the root vectors $xE_{12} + xE_{45}$ for the root α_1 satisfy

$$(xE_{12} + xE_{45})^2 = \begin{pmatrix} 0 & x & & & \\ & 0 & & & \\ & & 0 & & -x \\ & & & 0 & \\ & & & & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & & & & -x^2 \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \quad (xE_{12} + xE_{45})^3 = 0$$

$$\Rightarrow \quad \exp(xE_{12} + xE_{45}) = I + (xE_{12} + xE_{45}) + \frac{1}{2}(xE_{12} + xE_{45})^2 = \begin{pmatrix} 1 & x & & & -\frac{1}{2}x^2 \\ & 1 & & & \\ & & 1 & & -x \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

Indeed, setting

$$\begin{aligned} u_{\alpha_1} : \mathbb{G}_a &\to \mathrm{SO}_5(K) \\ x &\mapsto \begin{pmatrix} 1 & x & & & -\frac{1}{2}x^2 \\ & 1 & & & \\ & & 1 & & -x \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \end{aligned}$$

we have $u_{\alpha_1}(x)u_{\alpha_1}(y) = u_{\alpha_1}(x + y)$ and

$$tu_{\alpha_1}(x)t^{-1} = \begin{pmatrix} t_1 & & & & \\ & t_2 & & & \\ & & 1 & & \\ & & & t_2^{-1} & \\ & & & & t_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & x & & & -\frac{1}{2}x^2 \\ & 1 & & & \\ & & 1 & & -x \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} t_1^{-1} & & & & \\ & t_2^{-1} & & & \\ & & 1 & & \\ & & & t_2 & \\ & & & & t_1 \end{pmatrix} = \begin{pmatrix} 1 & t_1x & & & -\frac{1}{2}t_1^2x^2 \\ & 1 & & & \\ & & 1 & & -t_1x \\ & & & 1 & \\ & & & & 1 \end{pmatrix} = u_{\alpha_1}(\alpha_1(t)x),$$

as desired. Table 3 summarizes the information for all roots.

Figures 5 and 6 show the root system and spherical apartment of $\mathrm{SO}_5(K)$. Note that the root system of $\mathrm{SO}_5(K)$ is isomorphic to that of $\mathrm{Sp}_4(K)$ from Figure 3 by a 45° rotation. This exceptional isomorphism of root systems $\mathsf{B}_2 \cong \mathsf{C}_2$ is reflected in the isomorphism of Lie algebras $\mathfrak{sp}_4(K) \cong \mathfrak{so}_5(K)$.

As expected given this observation, the calculation of the Weyl group of $\mathrm{SO}_5(K)$ is similar to that for $\mathrm{Sp}_4(K)$ and also yields the dihedral group of order 8. One then verifies as in $\mathrm{Sp}_4(K)$ that Figures 5 and 6 show $\mathbb{R} \otimes X^*$ and $\mathbb{R} \otimes X_*$, respectively, under the unique (up to scalar) admissible

α	$\mathfrak{g}_\alpha$	U_α	$-\alpha$	$\mathfrak{g}_{-\alpha}$	$U_{-\alpha}$
$\alpha_1+\alpha_2$	$\begin{pmatrix} 0 & & & x & \\ & 0 & & & x \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & x & \\ & 1 & & & x \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$	$-\alpha_1-\alpha_2$	$\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ x & & & 0 & \\ & x & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ x & & & 1 & \\ & x & & & 1 \end{pmatrix}$
$\alpha_1-\alpha_2$	$\begin{pmatrix} 0 & x & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & x \\ & & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & x & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & x \\ & & & & 1 \end{pmatrix}$	$\alpha_2-\alpha_1$	$\begin{pmatrix} 0 & & & & \\ x & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & x & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & \\ x & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & x & 1 \end{pmatrix}$
α_1	$\begin{pmatrix} 0 & & x & & \\ & 0 & & & \\ & & 0 & & -x \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & x & & -\frac{1}{2}x^2 \\ & 1 & & & \\ & & 1 & & -x \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$	$-\alpha_1$	$\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ x & & 0 & & \\ & & & 0 & \\ & & -x & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ x & & 1 & & \\ & & & 1 & \\ -\frac{1}{2}x^2 & & -x & & 1 \end{pmatrix}$
α_2	$\begin{pmatrix} 0 & & & & \\ & 0 & x & & \\ & & 0 & x & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & \\ & 1 & x & \frac{1}{2}x^2 & \\ & & 1 & x & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$	$-\alpha_2$	$\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & x & 0 & & \\ & & x & 0 & \\ & & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & x & 1 & & \\ & \frac{1}{2}x^2 & x & 1 & \\ & & & & 1 \end{pmatrix}$

Table 3: Roots, root spaces, and one-dimensional unipotent subgroups of $\mathrm{SO}_5(K)$

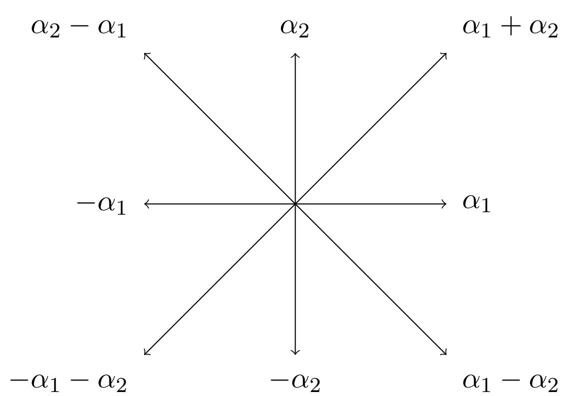

Figure 5: Root system of $\mathrm{SO}_5(K)$

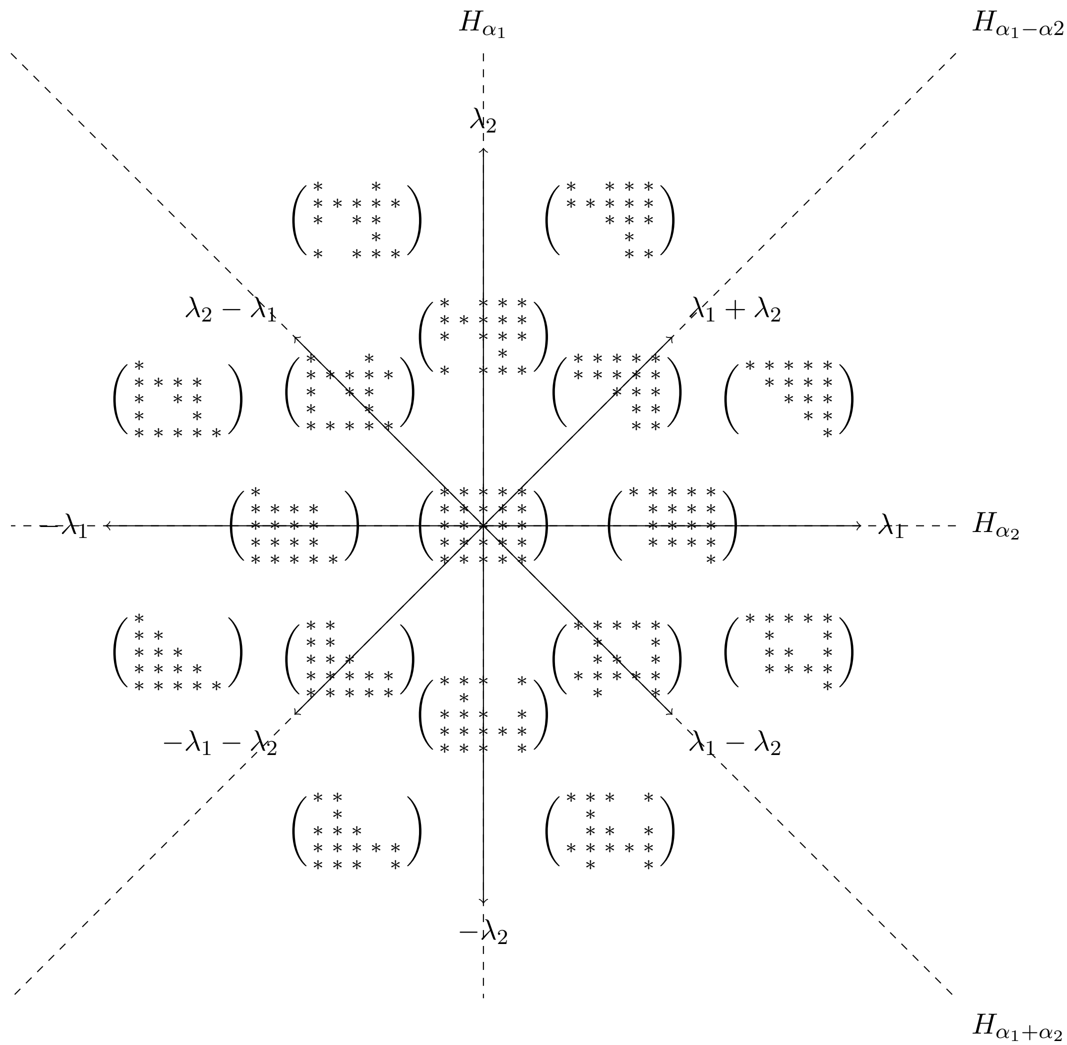

Figure 6: Coroots, spherical apartment, and associated parabolic subgroups $\mathrm{SO}_5(K)$

inner product, and that the Weyl group acts compatibly on $\mathbb{R} \otimes X_*$ and on the overlaid parabolic subgroups. Recall also that the root and weight lattices are determined up to isomorphism by the root system, hence isomorphic to those of $\mathrm{Sp}_4(K)$. Indeed, we have $\alpha_i^\vee = 2\lambda_i$ and $(\pm\alpha_i \pm \alpha_j)^\vee = \pm\lambda_i \pm \lambda_j$, so

$$\Phi^\vee = \{\pm 2\lambda_1, \pm 2\lambda_2, \pm\lambda_1 \pm \lambda_2\} = \{\pm\lambda_i \pm \lambda_j : 1 \leq i, j \leq 2\},$$

allowing $i = j$, and the last expression holds for any $\mathrm{SO}_{2n+1}(K)$ with $1 \leq i, j \leq n$. The root lattice Q is the span of $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_1 \pm \alpha_2\}$. For $\alpha = c_1\alpha_1 + c_2\alpha_2 \in \mathbb{R} \otimes X^*, c_i \in \mathbb{R}$, we have $\alpha \in P$ if and only if $\langle \alpha, \pm 2\lambda_i \rangle = \pm 2c_i$ and $\langle \alpha, \pm\lambda_i \pm \lambda_j \rangle = \pm c_i \pm c_j$ are integral, so

$$X^*(T) = Q = \{c_1\alpha_1 + c_2\alpha_2 : c_i \in \mathbb{Z}\}$$
$$P = \{c_1\alpha_1 + c_2\alpha_2 : c_i \in \frac{\mathbb{Z}}{2}, c_1 + c_2 \in \mathbb{Z}\}.$$

Thus $\mathrm{SO}_5(K)$ is adjoint. Alternatively, for $n = 2$, the isomorphism $\mathfrak{sp}_4(K) \cong \mathfrak{so}_5(K)$ allows one to read off Q and P from the corresponding lattices for $\mathrm{Sp}_4(K)$ found in the previous section.

The simply-connected semisimple group of type B_n is called $\mathrm{Spin}_{2n+1}(K)$. For $n = 2$, the isomorphism $\mathfrak{sp}_4(K) \cong \mathfrak{so}_5(K)$ allows one to construct an isogeny (in fact a double cover) $\mathrm{Sp}_4(K) \to \mathrm{SO}_5(K)$, and $\mathrm{Sp}_4(K)$ being simply-connected can be identified with $\mathrm{Spin}_5(K)$. This depends on the exceptional isomorphism $\mathsf{B}_2 \cong \mathsf{C}_2$ and does not generalize to arbitrary rank.

5 General Theory over Non-Algebraically Closed Fields

The remainder of this paper briefly addresses the relative structure theory of reductive groups, that is, when non-algebraically closed fields of definition are considered. We only outline the main ideas and give few details and no proof. Throughout, let k be a subfield of the algebraically closed field K. Again, the reader is encouraged to read the general theory together with the explicit examples in Section 6.

5.1 Field of Definition and k-Groups

Recall that every linear algebraic group can be realized as a closed subgroup of some $\mathrm{GL}_n(K)$. We are now interested in subgroups given by polynomials with coefficients in k. Some care is needed here. A set $X \subset \mathbb{A}^m$ is called *k-closed* if it is the set of zeros of some polynomials with coefficient in k, generating an ideal $I \subset K[T_1, \ldots, T_m]$. More intrinsic, however, is $\mathcal{I}(X) = \sqrt{I}$, which may properly contain I. One says that X is *defined over* k if $\mathcal{I}(X)$ is generated by polynomials with coefficients in k, or equivalently by $\mathcal{I}_k(X) := \mathcal{I}(X) \cap k[T_1, \ldots, T_m]$. In this case, we have $K[X] = K \otimes_k k[x]$ for the k-algebra $k[x] := k[T]/\mathcal{I}_k(X)$.

This extension of scalar suggests the following more intrinsic treatment of field of definition. Given a K-vector space V, a *k-structure* on V is a k-subspace V_k such that $V \cong K \otimes_k V_k$. This means that some K-basis for V is also a k-basis for V_k. A *k-structure* on a K-algebra A is a k-subalgebra A_k that is a k-structure on the underlying vector space of A. An ideal I of A is *defined* *over k* if $I_k := I \cap A_k$ is a k-structure on I, or equivalently, if I_k generates I as an ideal. Note that this agrees with the definition above. Without entering into details, an *affine k-variety* X is an affine K-variety together with a k-structure on the coordinate ring; this defines a *k-topology* on X, compatible k-structures on coordinate rings of subvarieties, and the *k-rational points* $X(k)$ of X. These definitions can then be generalized to arbitrary varieties or schemes; for details, see Sections AG11, 12 of [1].[12]

A homomorphism $f : A \to B$ of K-algebras with k-structures A_k and B_k is *defined over k* or is a *k-morphism* if $f(A_k) \subset B_k$, i.e. if they respect the k-structure. A morphism of k-varieties is *defined over k* or is a *k-morphism* if the induced map on the coordinate ring is defined over k, which sets up a contravariant equivalence between the category of affine k-varieties under k-morphisms and the category of affine K-algebras with k-structures under k-morphisms. The product of k-varieties has a natural k-structure and is again a k-variety. An algebraic group G whose underlying variety is a k-variety is *defined over k* or is a *k-group* if multiplication and inversion are defined over k. An *affine* or *linear k-group* is one whose underlying variety is affine. For a linear k-group G, the underlying K-variety may be embedded as a closed subgroup of some $\mathrm{GL}_n(K)$ in such a way that the subgroup $G(k)$ of G equals $G \cap \mathrm{GL}_n(k)$. This concrete view will suffice for our discussion.

A *k-torus* is a torus defined over k. A k-torus T is called *k-split* if $X(T)_k$, the group characters of T defined over k, spans $k[T]$, or equivalently if T is k-isomorphic to some $\mathbb{G}_m^n$, that is, isomorphic under a k-morphism. A k-torus T is called *k-anisotropic* if $X(T)_k = 0$.

5.2 Combinatorial Data of Reductive k-Groups

Let G now denote a reductive k-group. A *maximal k-split torus* of G is a k-split torus maximal under containment. It can be shown that maximal k-split tori of G are conjugate under $G(k)$, so we may define the *k-rank* of G to be the dimension of any such torus. The relative structure theory of reductive groups uses combinatorial data analogous to those in the absolute theory. However, the role of the maximal torus and Borel subgroups in the absolute theory are now played by the maximal k-split torus and minimal parabolic k-subgroups; in fact, G may have no Borel k-subgroup. Thus, for example, the conjugacy theorem states that minimal parabolic subgroups defined over k are conjugate under $G(k)$. Many of the arguments in the relative theory depend on a careful analysis of these parabolic subgroups.

Fix a maximal k-split torus S. The set of roots of G relative to the adjoint action of S is denoted ${}_k\Phi$, and its elements are called *k-roots*. This gives a root space decomposition of $\mathfrak{g}$. The *k-Weyl group* ${}_kW = N_G(S)/C_G(S)$, which is finite, acts simply transitively on the set of minimal parabolic k-subgroups containing $C_G(S)$. It also acts faithfully on $X^*(S)$ and $X_*(S)$, and by linear extension on $\mathbb{R} \otimes X^*(S)$ and $\mathbb{R} \otimes X_*(S)$. These vector spaces can be endowed with an *admissible* inner product, one invariant under ${}_kW$.

[12] Again we only consider reduced coordinate rings. Moreover, note that this definition of k-variety as a K-variety with k-structure is not intrinsic to k. For a more modern treatment of k-groups as functor of points of group schemes over k, see [7].

If G is semisimple, then ${}_k\Phi$ is a (possibly non-reduced) root system[13] in $\mathbb{R} \otimes X^*(S)$ of rank the k-rank of G and abstract Weyl group isomorphic to ${}_kW$. In particular, for each $\alpha \in {}_k\Phi$, ${}_kW$ contains an element σ_α that acts on $\mathbb{R} \otimes X^*(S)$ as a reflection relative to α, and these σ_α generate ${}_kW$. Under an admissible inner product on $\mathbb{R}\otimes X^*(S)$, these become orthogonal reflections. If P is a minimal parabolic k-subgroup containing S, then there exists a basis of ${}_k\Phi$ under which the positive roots are those k-roots occuring in $R_u(P)$. Moreover, $G(k)$ contains a full set of representatives of ${}_kW$, and we have the *relative Bruhat decomposition* $G(k) = \bigcup_{\sigma \in {}_kW} U(k)\sigma U(k)$.

For G merely reductive, $S' := (S \cap (G,G))^\circ$ is a maximal k-split torus of the semisimple derived group (G,G). With this choice of maximal k-split torus, ${}_k\Phi$, ${}_kW$, and other combinatorial data of G and (G,G) can be identified as in the absolute theory. The k-rank of (G,G) is the *semisimple k-rank* of G. Then ${}_k\Phi$ is a (possibly non-reduced) root system in its span in $\mathbb{R} \otimes X^*(S)$, which can be identified with $\mathbb{R} \otimes X^*(S')$, of rank the semisimple k-rank of G and abstract Weyl group isomorphic to ${}_kW$.

5.3 Classification of Reductive k-Groups

A reductive k-group G is called *k-split* if G has a maximal k-split torus, i.e. if its k-rank equals its (absolute) rank. In this case, the maximal k-split torus S is in fact a maximal torus T, and the relative and absolute combinatorial data coincide. For example, $\mathrm{GL}_n(K)$, $\mathrm{SL}_n(K)$, $\mathrm{PSL}_n(K)$, $\mathrm{Sp}_n(K)$, $\mathrm{PSp}_n(K)$ are all split over the prime field of K, and the structure theory of $G(k)$ is therefore in each case quite similar to that of G. In particular, the root spaces are one-dimensional.

New phenomena occur for non-split groups. For example, suppose the maximal k-split torus S is (properly) contained in a maximal torus T. Since some roots that are distinct relative to T can become equal relative to S, the corresponding root spaces merge and need no longer be one-dimensional. Moreover, such data as ${}_k\Phi$ and the k-rank are not preserved under arbitrary isogenies. One recovers their invariance by restricting to central isogenies; see for example Section 22 of [1].

Note that a k-group may trivially be viewed as a K-group, which is automatically split, by ignoring the rational structure. Since the absolute theory already classifies reductive K-groups, the classification of reductive k-groups comes down to determining, for each reductive K-group G representing a K-isomorphism class, the k-groups up to k-isomorphism that become isomorphic to G over K. A central tool for this is Galois cohomology, the group cohomology for the absolute Galois group of k; it can be shown that k-groups that become isomorphic over K are classified up to k-isomorphism by a certain first cohomology set. For G nonabelian, this set is not a group but a pointed set with a distinguished "identity" that corresponds in this classification to the k-split form; non-identity classes correspond to the various non-split forms, obtained by "twisting" the split form. Introductions to Galois cohomology can be found for example in [13] and [11]. Some references on the Galois cohomology of algebraic groups are Chapter VI Section 1 of [7], Chapter 6 of [8], Section 29 of [5], and [6].

[13] Even for a non-reduced root system, it follows quickly from the integrality condition that $\pm\alpha, \pm 2\alpha$ are the only possible forms of roots that are multiples of one another. One can show that only one new infinitely family of irreducible root systems arises, called type BC_l because it is a union of B_l and C_l with some roots identified.

Note that the relative combinatorial data introduced above has no content if G is k-*anisotropic*, that is, contains no nontrivial k-split torus. More generally, the data ignores what is called the *anisotropic kernel* of G. Tits's classification theorem for k-reductive groups states that the k-isomorphism class of G is determined by its K-isomorphism class, its anisotropic kernel, and an additional data called the *index*; in particular, the absolute theory already essentially gives the classification of split reductive k-groups. Given the classification over K, it remains to determine for each K-isomorphism class the possible anisotropic kernels and indices, a work started by Tits and continued by Selbach. The classification of anisotropic kernels depends heavily on the nature of the field k, and in particular on its Brauer group; for details, see Chapter VI of [7].

6 Examples: Orthogonal Groups over $\mathbb{R}$

As already mentioned, $\mathrm{GL}_3(\mathbb{C})$, $\mathrm{SL}_3(\mathbb{C})$, $\mathrm{PSL}_3(\mathbb{C})$, $\mathrm{Sp}_4(\mathbb{C})$, and $\mathrm{PSp}_4(\mathbb{C})$ are split over the base field $\mathbb{Q}$, so the computations of combinatorial data are analogous for their $\mathbb{R}$-rational points. Here we consider (nondegenerate) symmetric bilinear forms and orthogonal groups, which, unlike alternating bilinear forms, are in general not unique up to equivalence over non-algebraically closed fields.

6.1 Generalities on Symmetric Bilinear Forms and Orthogonal Groups

Let the pair (V, B) denote a vector space V equipped with a nondegenerate symmetric bilinear form B. More details on the basic notions of bilinear forms (or on the equivalent data of quadratic forms for characteristic $\neq 2$) can be found for example in Chapter IV of [10] and I.18 of [7].

Let $V = K^n$ with k-structure $V_k = k^n$, and let E be an extension of k lying in K. A given (V_k, B_k) extends uniquely to (V_E, B_E) by extension of scalars. Recall that V has a unique (up to equivalence) nondegenerate symmetric bilinear form, so that any $(V_k, B_k), (V_k, B'_k)$ become equivalent under extension to K and there exists up to isomorphism a unique rank-n orthogonal group over K. Like k-forms of algebraic groups, and in fact by the same argument, equivalence classes of pairs (V_k, B_k) are also classified by a certain first cohomology set; for details, see I.19 of [7] or III Appendix 2 §2 of [11]. Mapping (V_k, B_k) to its isometry group therefore induces a map from the first cohomology set classifying equivalence classes of nondegenerate symmetric bilinear forms on V_k to the first cohomology set classifying k-isomorphism classes of rank-n orthogonal groups over k. While it can be shown that this is surjective, it may fail to be injective, as inequivalent symmetric bilinear forms on V_k can have k-isomorphic isometry groups; for example, B_k and $-B_k$ have the same isometry group but may be inequivalent.

Recall that the k-split orthogonal group is naturally identified with the distinguished "identity" in the first cohomolgy set. We now define the corresponding notion of a split symmetric bilinear form. A nonzero $v \in V$ is called *isotropic* if $B(v, v) = 0$. We call (V, B) *totally isotropic*, *isotropic*, or *anisotropic* if correspondingly all, some, or no nonzero vector is isotropic. A *hyperbolic plane* is a two-dimensional vector space V with a nondegenerate isotropic symmetric bilinear form B. It follows easily that there are two linearly independent isotropic vectors $v, w \in V$ with $B(v, w) \neq 0$, so that a hyperbolic plane over a given field is uniquely determined up to isometry. Concretely, on

$V = k^2$, we can for example take the bilinear form given by $\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$, with isotropic vectors $(1, \pm 1)$, or the equivalent bilinear form given by $\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$, with isotropic vectors $(1, 0)$ and $(0, 1)$.

The *isotropy index* or *Witt index* of (V, B) is the maximum dimension of totally isotropic subspaces of V. Witt's theorem states that (V, B) of Witt index m is equivalent to an orthogonal sum of m hyperbolic planes and an anisotropic complement, the *anisotropic kernel* of (V, B), that is unique up to isometry. A pair (V, B) is called *split* if it has the largest possible Witt index $\lfloor \dim V/2 \rfloor$, or equivalently by Witt's theorem, if it is an orthogonal sum of hyperbolic planes and, in odd dimension, a one-dimensional anisotropic kernel.

With this definition, it can be shown that (V_k, B_k) is split if and only if the orthogonal group over k that it defines is k-split; that is, in the above map between first cohomology sets, the preimage of the identity is exactly the classes corresponding to the k-split forms. Moreover, we say that E *splits* (or is a *splitting field* of) (V_k, B_k) if (V_E, B_E) is split, and that E *splits* (or is a *splitting field* of) a k-group G if its base extension to E is E-split. For any E, we then conclude that E splits (V_k, B_k) if and only if it splits the orthogonal group over k that it defines.

6.2 Orthogonal Groups over $\mathbb{R}$

Recall that every (nondegenerate) symmetric bilinear form has an orthogonal basis and that scaling a basis vector multiplies the corresponding diagonal entry by a square. Over $\mathbb{R}$, each diagonal entry can thus be made ± 1. Sylvester's theorem for real symmetric bilinear forms states that the numbers m and k of $+1$ and -1 is independent of the choice of orthogonal basis. Nondegenerate real symmetric bilinear forms are therefore classified up to equivalence by this pair (m, k), called the *index* of the form. Write $SO(m, k)$ for the special orthogonal group over $\mathbb{R}$ preserving a bilinear form with index (m, k).

Here we consider the distinct $\mathbb{R}$-forms of special orthogonal groups in $\mathbb{R}^5$, which all become isomorphic over the algebraic closure $\mathbb{C}$ to the unique special orthogonal group $\mathrm{SO}_5(\mathbb{C})$ already examined in Section 4.6. Since $SO(m, k) = SO(k, m)$, we only have three indices to consider: $(5, 0)$ (the usual dot product), $(4, 1)$, and $(3, 2)$. The corresponding isometry groups give all the $\mathbb{R}$-forms up to $\mathbb{R}$-isomorphism.

6.2.1 Split $\mathbb{R}$-Form: $SO(3, 2)$

The space $\mathbb{R}^5$ with bilinear form of index $(3, 2)$ given by $\mathrm{diag}(1, 1, 1, -1, -1)$ with respect to the standard basis has Witt index 2, being the orthogonal sum of two hyperbolic planes, $\mathbb{R}e_1 \oplus \mathbb{R}e_5$ and $\mathbb{R}e_2 \oplus \mathbb{R}e_4$ with the restricted bilinear form given by $\mathrm{diag}(1, -1)$, and an isotropic kernel, the line $\mathbb{R}e_3$. Changing bases for the hyperbolic planes so that the bilinear forms are given by $\mathrm{antidiag}(1, 1)$ and $\mathrm{antidiag}(-1, -1)$, we get

$$M = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}.$$

A diagonal matrix t lies in $O(3,2)$ if and only if $tMt = M$. Since

$$tMt = \begin{pmatrix} & & & & t_1t_5 \\ & & & -t_2t_4 & \\ & & t_3^2 & & \\ & -t_4t_2 & & & \\ t_5t_1 & & & & \end{pmatrix}$$

this occurs exactly when $t_1t_5 = t_2t_4 = t_3^2 = 1$. Then $t \in SO(3,2)$ if and only if $1 = \det t = (t_1t_5)(t_2t_4)t_3 = t_3$, so

$$T := \mathrm{SO}(3,2) \cap D = \left\{ \begin{pmatrix} t_1 & & & & \\ & t_2 & & & \\ & & 1 & & \\ & & & t_2^{-1} & \\ & & & & t_1^{-1} \end{pmatrix} : t_1, t_2 \in \mathbb{R}^* \right\}.$$

This is in fact not only a maximal $\mathbb{R}$-split torus but also an (absolute) maximal torus, so $SO(3,2)$ is $\mathbb{R}$-split. The relative combinatorial data therefore reduce to the absolute case, $\mathrm{SO}_5(\mathbb{C})$, already considered in Section 4.6. The calculations for the Lie algebra, roots, and the corresponding root spaces and one-dimensional unipotent subgroups also remain the same; see Table 3 and Figures 5 and 6.

6.2.2 $SO(4,1)$

The Witt index is now 1, and by a similar change of basis, we may use the bilinear form given by

$$M = \begin{pmatrix} & & & & 1 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & & & & \end{pmatrix}.$$

By a similar calculation,

$$S := \mathrm{SO}(4,1) \cap D = \left\{ \begin{pmatrix} t_1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & t_1^{-1} \end{pmatrix} : t_1 \in \mathbb{R}^* \right\},$$

and this is a maximal $\mathbb{R}$-split torus. Thus $SO(4,1)$ has $\mathbb{R}$-rank 1 and $X^*(S)$ is free abelian on $\alpha_1 : t \mapsto t_1$. It is easy seen that

$$C_G(S) = \begin{pmatrix} * & & & & \\ & * & * & * & \\ & * & * & * & \\ & * & * & * & \\ & & & & * \end{pmatrix}, \quad N_G(S) = \begin{pmatrix} * & & & & \\ & * & * & * & \\ & * & * & * & \\ & * & * & * & \\ & & & & * \end{pmatrix} \cup \begin{pmatrix} & & & & * \\ & * & * & * & \\ & * & * & * & \\ & * & * & * & \\ * & & & & \end{pmatrix},$$

so the $\mathbb{R}$-Weyl group is ${}_{\mathbb{R}}W = \{\mathrm{id}, \sigma\}$, with σ represented by

$$n = \begin{pmatrix} & & & & 1 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & & & & \end{pmatrix}.$$

Here and in the sequel, note that the anisotropic kernel of the bilinear form is irrelevant for the combinatorial data.

Computing $A^t M + MA = 0$ shows that the Lie algebra consists of matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{21} & 0 & a_{23} & a_{24} & -a_{12} \\ a_{31} & -a_{23} & 0 & a_{34} & -a_{13} \\ a_{41} & -a_{24} & -a_{34} & 0 & -a_{14} \\ 0 & -a_{21} & -a_{31} & -a_{41} & -a_{11} \end{pmatrix},$$

from which one easily finds the roots and associated data, shown in Table 4.

$\mathfrak{g}_{\alpha_1,i}$	$U_{\alpha_1,i}$	$\mathfrak{g}_{-\alpha_1,i}$	$U_{-\alpha_1,i}$
$\mathfrak{g}_{\alpha_1,1} = \begin{pmatrix} 0 & x & & & \\ & 0 & & & -x \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$	$U_{\alpha_1,1} = \begin{pmatrix} 1 & x & & & -\frac{1}{2}x^2 \\ & 1 & & & -x \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$	$\mathfrak{g}_{-\alpha_1,1} = \begin{pmatrix} 0 & & & & \\ x & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & -x & & & 0 \end{pmatrix}$	$U_{-\alpha_1,1} = \begin{pmatrix} 1 & & & & \\ x & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ -\frac{1}{2}x^2 & -x & & & 1 \end{pmatrix}$
$\mathfrak{g}_{\alpha_1,2} = \begin{pmatrix} 0 & & x & & \\ & 0 & & & \\ & & 0 & & -x \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$	$U_{\alpha_1,2} = \begin{pmatrix} 1 & & x & & -\frac{1}{2}x^2 \\ & 1 & & & \\ & & 1 & & -x \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$	$\mathfrak{g}_{-\alpha_1,2} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ x & & 0 & & \\ & & & 0 & \\ & & -x & & 0 \end{pmatrix}$	$U_{-\alpha_1,2} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ x & & 1 & & \\ & & & 1 & \\ -\frac{1}{2}x^2 & & -x & & 1 \end{pmatrix}$
$\mathfrak{g}_{\alpha_1,3} = \begin{pmatrix} 0 & & & x & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & -x \\ & & & & 0 \end{pmatrix}$	$U_{\alpha_1,3} = \begin{pmatrix} 1 & & & x & -\frac{1}{2}x^2 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & -x \\ & & & & 1 \end{pmatrix}$	$\mathfrak{g}_{-\alpha_1,3} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ x & & & 0 & \\ & & & -x & 0 \end{pmatrix}$	$U_{-\alpha_1,3} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ x & & & 1 & \\ -\frac{1}{2}x^2 & & & -x & 1 \end{pmatrix}$

Table 4: Roots, independent root vectors, and one-dimensional unipotent subgroups of $\mathrm{SO}(4,1)$

In particular, ${}_{\mathbb{R}}\Phi = \{\pm\alpha_1\}$ is irreducible of type A_1, with both rootspaces of dimension greater than 1 (in fact 3); we denoted the spaces spanned by the natural independent root vectors $\mathfrak{g}_{\pm\alpha_1,i}$ and the corresponding one-dimensional unipotent subgroups $U_{\pm\alpha_1,i}$, $1 \leq i \leq 3$. Moreover, there are nondiagonal generalized eigenvectors of the zero character,

$$\begin{pmatrix} 0 & & & & \\ & 0 & x & & \\ & -x & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ & 0 & & x & \\ & & 0 & & \\ & -x & & 0 & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & x & \\ & & -x & 0 & \\ & & & & 0 \end{pmatrix},$$

whose exponentials generate

$$A = \left\{ \begin{pmatrix} 1 & & \\ & O & \\ & & 1 \end{pmatrix} : O \in \mathrm{SO}_3(\mathbb{R}) \right\}$$

corresponding to the anisotropic kernel of the bilinear form. Recall that neither of these phenomena occurs in the absolute theory.

The absolute theory associated a parabolic subgroup $P_\lambda = \langle T, U_\alpha : \alpha \in \Phi, \langle \alpha, \lambda \rangle \geq 0 \rangle$ to λ, and in fact to each facet, in the coroot space. The relative theory has an analogous concrete interpretation of the parabolic subgroups. Here, the relative coroot space $\mathbb{R} \otimes X_*(S)$ has one hyperplane H_{α_1} dividing it into two maximal facets, containing λ_1 and $-\lambda_1$. Note that an absolute maximal torus T is the group generated by the exponentials of the generalized eigenvectors for the zero character. Similarly, we define P_{λ_1} to be the group generated by S, $U_{\alpha_1,i}$, and additionally A, and similarly for $P_{-\lambda_1}$:

$$P_{\lambda_1} = \langle S, A, U_{\alpha_1,i} : 1 \leq i \leq 3 \rangle = \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \\ & & & & * \end{pmatrix}$$

$$P_{-\lambda_1} = \langle S, A, U_{-\alpha_1,i} : 1 \leq i \leq 3 \rangle = \begin{pmatrix} * & & & & \\ * & * & * & * & \\ * & * & * & * & \\ * & * & * & * & \\ * & * & * & * & * \end{pmatrix}$$

The $\mathbb{R}$-coroot lattice encodes the containment of $\mathbb{R}$-parabolic subgroups, so that maximal facets again correspond to minimal $\mathbb{R}$-parabolic subgroups. Unlike in the absolute theory, however, they may no longer be Borel subgroups, as this example demonstrates. However, like in the absolute theory, ${}_{\mathbb{R}}W$ acts simply transitively on the set of minimal $\mathbb{R}$-parabolic subgroups; σ (acting as conjugation by n) interchanges $P_{\pm\lambda_1}$, compatible with its action on $\mathbb{R} \otimes X_*(S)$ as a reflection relative to λ_1.

6.2.3 Anisotropic $\mathbb{R}$-Form: $SO(5,0)$

The bilinear form of index $(5,0)$ is positive definite, say given by $M = I$. This is just the dot product, so $SO(5,0)$ is the usual real special orthogonal group $SO_5(\mathbb{R})$. With this choice of M, a similar calculation as above shows that there is no nontrivial diagonal element satisfying $tMt = t$; in fact $SO(5,0)$ has no non-trivial $\mathbb{R}$-split torus, i.e. it is $\mathbb{R}$-anisotropic. The relative combinatorial data therefore have no content.

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