DIAGRAMMATICS FOR EXT-ENHANCED SOERGEL BIMODULES IN TYPE A_1

SHOTARO MAKISUMI

Abstract. This unfinished note gives a diagrammatic presentation for the monoidal category of "Ext-enhanced" Soergel bimodules in type A_1.

1. Introduction

1.1. **About this document.** Most of this note was written in Fall 2018 while I was preparing for the joint work [HM] with M. Hogancamp. Since that paper ended up only using the GL_2-realization, where most of the necessary calculations were already available in [GH, §3.5], the generality explored in the present note was not needed, and I never returned to it. I am making it available in that half-finished state in 2022 only for ease of reference.

In its current state, this note was never meant to be seen by anyone beyond interested collaborators. I have not always clearly defined the setup, terms, or notation used. I expect that anyone who is interested in this material in the first place will have the necessary background and context to be able to read this note, but I apologize in advance for these omissions.

1.2. **What is done here.** Recall the usual setting of the theory of Soergel bimodules. Let $\mathfrak{h}$ be a realization of a Coxeter system W, and let $R = \mathrm{Sym}(\mathfrak{h}^*)$ be the symmetric algebra with the grading $\deg \mathfrak{h}^* = 2$. Then the associated category of Soergel bimodules $\mathrm{SBim}(\mathfrak{h}, W)$ is a full subcategory of the abelian category $R\text{-}\mathsf{gmod}\text{-}R$ of graded R-bimodules.

Now, consider the bounded derived category $D^{\mathrm{b}}(R\text{-}\mathsf{gmod}\text{-}R)$, which is monoidal under the derived tensor product $\otimes_R^L$. We view $\mathrm{SBim}(\mathfrak{h}, W)$ as a full subcategory of $D^{\mathrm{b}}(R\text{-}\mathsf{gmod}\text{-}R)$ of complexes supported in cohomological degree 0.

Definition 1.1. The category of *Ext-enhanced Soergel bimodules* $\mathrm{SBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ is the smallest strictly full monoidal additive subcategory of $D^{\mathrm{b}}(R\text{-}\mathsf{gmod}\text{-}R)$ containing SBim and closed under $[m]$, $m \in \mathbb{Z}$.

In other words, objects of $\mathrm{SBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ are finite direct sums of objects of the form $B[m]$, where B is a Soergel bimodule and $m \in \mathbb{Z}$. For two such objects,

$$\mathrm{Hom}_{\mathrm{SBim}^{\mathrm{Ext}}(\mathfrak{h},W)}(B[m], B'[m']) = \mathrm{Ext}^{m'-m}_{R\text{-}\mathsf{gmod}\text{-}R}(B, B').$$

The goal of this document is to give a monoidal presentation for $\mathrm{SBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ in the case that $W = S_2$, extending the monoidal presentation for $\mathrm{SBim}(\mathfrak{h}, W)$ by Elias–Khovanov.

As in Elias–Khovanov [EK], we will actually give a monoidal presentation for a Bott–Samelson category $\mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$, whose morphism spaces are bigraded

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(by cohomological grading and bimodule grading), from which $\mathrm{SBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ can be recovered up to equivalence in a formal way. For this Bott–Samelson category, we need some care in handling the interaction between the cohomological grading and the monoidal structure. In fact, $\mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ will not be a monoidal category, but rather "supermonoidal" in the cohomological grading. This is carefully explained in §3.

Correspondingly, in §4, starting with an arbitrary realization $(\mathfrak{h}, W)$ of $W = S_2$ satisfying Demazure surjectivity, we will define $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ as a strict supermonoidal category with an additional grading via a diagrammatic presentation. Under certain assumptions, we will define a supermonoidal functor $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W) \to \mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ that is an equivalence when $(\mathfrak{h}, W)$ is a Soergel realization.

1.3. **Motivation and relation to previous work.** In the case of the GL_2 realization, the Ext-enhanced diagrammatic category was introduced in [HM] and used to provide evidence for a triply-graded version of the so-called monoidal Koszul duality. See the introduction to that paper for the motivation to study Ext-enhanced Soergel bimodules. One may also speculate that this triply-graded Koszul duality is connected to a duality for character sheaves.

1.4. **Contents.** §2 contains some algebraic preliminaries. In §3, we will assume in addition that $\Bbbk$ is a field and study the algebraic category of Ext-enhanced Bott–Samelson bimodules. In §4, we define a diagrammatic version of this category (now for $\Bbbk$ a domain). In §5, we assume that $\Bbbk$ is a field, and we prove our main result, that the diagrammatic category is equivalent to the bimodule category.

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2. Algebraic preliminary

The following data will be fixed throughout this paper. Let (W, S) be a Coxeter system. Let $\Bbbk$ be a domain [some arguments in this section currently assume that $\Bbbk$ is a field], and let

$$\mathfrak{h} = (V, \{\alpha_s^\vee\}_{s\in S} \subset V, \{\alpha_s\}_{s\in S} \subset V^*)$$

be a realization of (W, S) over $\Bbbk$, in the sense of [EW]. We assume throughout that $\mathfrak{h}$ is balanced and satisfies Demazure surjectivity.

2.1. **Some bigraded algebras and dgg modules.** All our algebras and modules will be bigraded with cohomological grading and Soergel grading. The shifts [1] and (1) shift the cohomological grading and the Soergel grading, respectively, down by 1. It will be useful to introduce the combine shift $[\![1]\!] := [1](-2)$.

Define the following bigraded $\Bbbk$-algebras.

$$R := \mathrm{Sym}^\bullet(V^*(-2)),$$
$$\Lambda := \Lambda^\bullet(V^*[\![1]\!]),$$
$$\Lambda^\vee := \Lambda^\bullet(V[\![-1]\!]).$$

We will frequently write $R^e = R \otimes R$.

Let $s \in S$. Define

$$\Lambda^s := \Lambda^\bullet((V^*)^s[\![1]\!]) \hookrightarrow \Lambda, \tag{2.1}$$

and dually,

$$\Lambda^\vee \twoheadrightarrow \Lambda_s^\vee := \Lambda^\bullet((V/\Bbbk\alpha_s^\vee)[\![-1]\!]). \tag{2.2}$$

The inclusion $(V^*)^s \hookrightarrow V^*$ dualizes to the surjection $V \cong (V^*)^* \twoheadrightarrow ((V^*)^s)^*$ sending $v \mapsto \langle v, - \rangle|_{(V^*)^s}$. Since $\langle \alpha_s^\vee, - \rangle$ is 0 on $(V^*)^s$, we obtain a canonical isomorphism

$$V/\Bbbk\alpha_s^\vee \xrightarrow{\sim} ((V^*)^s)^*. \tag{2.3}$$

2.2. **Derivations on exterior algebras.** See [AMRW, §3.3]. Define

$$(-) \mathbin{\grave{\frown}} (-) : V[\![-1]\!] \otimes \Lambda \to \Lambda$$

by

$$v \mathbin{\grave{\frown}} (r_1 \wedge \cdots \wedge r_k) = \sum_{i=1}^{k} (-1)^{i+1} r_i(v) r_1 \wedge \cdots \wedge \widehat{r_i} \wedge \cdots \wedge r_k.$$

This induces

$$(-) \mathbin{\grave{\frown}} (-) : \Lambda^\vee \otimes \Lambda \to \Lambda. \tag{2.4}$$

Similarly, define

$$(-) \mathbin{\acute{\frown}} (-) : \Lambda \otimes \Lambda^\vee \to \Lambda^\vee.$$

It is straightforward to check that

$$s(r) = r - \alpha_s \wedge (\alpha_s^\vee \mathbin{\grave{\frown}} r), \tag{2.5}$$

$$s(x) = x - \alpha_s^\vee \wedge (\alpha_s \mathbin{\acute{\frown}} x). \tag{2.6}$$

for all $s \in S$, $r \in \Lambda$, $x \in \Lambda^\vee$.

Lemma 2.1. *Assume Demazure surjectivity. Then the map $\alpha_s^\vee \mathbin{\grave{\frown}} (-) : \Lambda \to \Lambda[\![1]\!]$ has kernel Λ^s and image $\Lambda^s[\![1]\!]$.*

Proof. Let ker and im be the kernel and the image. It follows from (2.5) that $\Lambda^s \subset \ker$. By Demazure surjectivity, there exists $\rho_s \in V^*$ such that $\langle \alpha_s^\vee, \rho_s \rangle = 1$. If $r \in \Lambda^s$, then

$$\alpha_s^\vee \mathbin{\grave{\frown}} (\rho_s \wedge r) = \langle \alpha_s^\vee, \rho_s \rangle r - \rho_s \wedge (\alpha_s^\vee \mathbin{\grave{\frown}} r) = r.$$

Thus $\Lambda^s[\![1]\!] \subset \mathrm{im}$. Now,

$$\begin{aligned}\binom{r}{k} &= \dim \Lambda^k V^* = \dim(\ker) + \dim(\mathrm{im}) \\ &\geq \dim \Lambda^k (V^*)^s + \dim \Lambda^{k-1}(V^*)^s = \binom{r-1}{k} + \binom{r-1}{k-1} = \binom{r}{k},\end{aligned}$$

so we must have equality throughout. Since $\Bbbk$ is a field, we are done. $\square$

Thus we obtain a short exact sequence

$$0 \longrightarrow \Lambda^s \longrightarrow \Lambda \xrightarrow{\alpha_s^\vee \grave{\frown}(-)} \Lambda^s[\![1]\!] \longrightarrow 0. \tag{2.7}$$

2.3. **Demazure operators on exterior algebras.** Let $s \in S$. Define the *exterior Demazure operator*

$$\partial_s : \Lambda^\vee \to \Lambda_s^\vee[\![-1]\!]$$

as the composition

$$\Lambda^\vee \xrightarrow{\alpha_s \frown (-)} \Lambda^\vee[\![-1]\!] \twoheadrightarrow \Lambda_s^\vee[\![-1]\!],$$

where the second arrow is the canonical surjection (2.2). Then ∂_s is the unique $\Bbbk$-linear endomorphism of $\Lambda^\vee$ satisfying the following two properties:

$$\partial_s(v) = \alpha_s(v) \qquad \text{for all } v \in V, \tag{2.8}$$

$$\partial_s(x \wedge y) = \partial_s(x) \wedge y + (-1)^{|x|} s(x) \wedge \partial_s(y) \qquad \text{for all } x, y \in \Lambda^\vee. \tag{2.9}$$

We refer to (2.9) as the *twisted Leibniz rule.* Uniqueness is clear, as is (2.8). For (2.9), note that $\alpha_s \frown (-)$ is a derivation hence satisfies the graded Leibniz rule, and $x - s(x) = 0$ in the quotient $\Lambda_s^\vee$ by (2.6).

Remark 2.2. Recall that, if $\Bbbk$ is a field of characteristic not equal to 2, then the classical Demazure (or divided difference) operator $\partial_s : R \to R(-2)$ may be defined as[1]

$$\partial_s(f) = \frac{f - s(f)}{\alpha_s}. \tag{2.10}$$

Equation (2.6) says that the exterior Demazure operator may be thought of as being given by a formula analogous to (2.10). Since $\alpha_s^\vee \wedge (-)$ kills $\Bbbk\alpha_s^\vee$, we should only expect $\partial_s(x)$ to be defined modulo $\Bbbk\alpha_s^\vee$.

Remark 2.3. There is no map $\partial_s : \Lambda^\vee \to \Lambda^\vee[\![-1]\!]$ satisfying (2.8) and (2.9). Indeed, these properties force

$$\partial_s(r_1 \wedge \cdots \wedge r_k) = \sum_{i=1}^{k} (-1)^{i+1} \alpha_s(r_i) s(r_1) \wedge \cdots \wedge s(r_{i-1}) \wedge \widehat{r_i} \wedge r_{i+1} \wedge \cdots \wedge r_k,$$

for all $k \geq 1$ and $r_1, \ldots, r_k \in V^*$, but already for $k = 3$, it is easy to check that the right hand side is not alternating, so that ∂_s is not well-defined.

The exterior Demazure operators will be used in the exterior forcing relation (4.7).

Assume Demazure surjectivity, and choose $\rho_s^\vee$ such that $\alpha_s(\rho_s^\vee) = 1$.

Lemma 2.4. *We have*

$$\Lambda^\vee = (\Lambda^\vee)^s \oplus (\rho_s^\vee \wedge (\Lambda^\vee)^s).$$

That is, any $x \in \Lambda^\vee$ can be uniquely written as

$$x = y + \rho_s^\vee \wedge z,$$

where $y, z \in (\Lambda^\vee)^s$.

Proof. Existence is clear because $V = V^s \oplus \Bbbk\rho_s^\vee$. For uniqueness, observe that given such an expression for x, we have $z = \alpha_s \frown x$ and $y = x - \rho_s^\vee \wedge z$. □

[1] See [EW, §3.3] for a definition in our generality (assuming only Demazure surjectivity).

3. Ext-enhanced Soergel bimodules

Let $\Bbbk$ be a commutative ring. Throughout, undecorated tensor products are over $\Bbbk$. Let V be a free $\Bbbk$-module of finite rank. Let $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$, and

$$R = \mathrm{Sym}_{\Bbbk}(V^*)$$

graded with $\deg V^* = 2$.

Let $R\text{-}\mathsf{gmod}\text{-}R$ be the category of graded R-bimodules and R-bimodule maps of degree 0.

3.1. **Complexes of graded R-bimodules.** We view a complex in $R\text{-}\mathsf{gmod}\text{-}R$ as a pair (M, d_M), where M and d_M are equipped with a $\mathbb{Z}$-grading

$$M = \bigoplus_i M^j, \qquad d_M = \bigoplus_j d_M^j : M \to M,$$

and each M^j is a graded R-bimodule, and $d_M^j : M^j \to M^{j+1}$ is an R-bimodule map of degree 0. For $m \in M^j$, we write $|m| = j$ and say that m is homogeneous of cohomological degree j.

Given two complexes M, N in $R\text{-}\mathsf{gmod}\text{-}R$, define the bigraded $\Bbbk$-module

$$\underline{\mathrm{Hom}}(M,N) = \bigoplus_{i,j} \underline{\mathrm{Hom}}^{i,j}(M,N) = \bigoplus_{i,j} \prod_p \mathrm{Hom}_{R\text{-}\mathsf{gmod}\text{-}R}(M^p, N^{p+j}(i)).$$

If $f \in \underline{\mathrm{Hom}}^{i,j}(M,N)$, we write $\deg f = (i,j)$ and say that f is homogeneous of degree (i,j). We also write $|f| = j$ and say that f has cohomological degree j. Equip $\underline{\mathrm{Hom}}(M,N)$ with the differential $d_{\underline{\mathrm{Hom}}(M,N)}$ of degree $(0,1)$ by

$$d_{\underline{\mathrm{Hom}}(M,N)}(f) = d_M \circ f - (-1)^{|f|} f \circ d_N.$$

It can be checked that $d_{\underline{\mathrm{Hom}}(M,N)} \circ d_{\underline{\mathrm{Hom}}(M,N)} = 0$. Define the total cohomology

$$\mathbb{H}\mathsf{om}(M,N) = H^{\bullet,\bullet}(\underline{\mathrm{Hom}}(M,N), d_{\underline{\mathrm{Hom}}(M,N)}).$$

Given three complexes L, M, N in $R\text{-}\mathsf{gmod}\text{-}R$, define the bigraded composition

$$- \circ - : \underline{\mathrm{Hom}}(M,N) \otimes \underline{\mathrm{Hom}}(L,M) \to \underline{\mathrm{Hom}}(L,N) \tag{3.1}$$

in the obvious way (involving no sign). This composition is associative in the obvious sense. Moreover, for $f \in \underline{\mathrm{Hom}}(M,N)$ and $g \in \underline{\mathrm{Hom}}(L,M)$, one checks directly that

$$d(f \circ g) = d(f) \circ g + (-1)^{|f|} f \circ d(g). \tag{3.2}$$

It follows from (3.2) that the composition (3.1) induces a bigraded associative composition on total cohomology:

$$- \circ - : \mathbb{H}\mathsf{om}(M,N) \otimes \mathbb{H}\mathsf{om}(L,M) \to \mathbb{H}\mathsf{om}(L,N). \tag{3.3}$$

Definition 3.1. The category $K^{\mathrm{b,bigr}}(R\text{-}\mathsf{gmod}\text{-}R)$ is defined as follows. Its objects are bounded complexes in $R\text{-}\mathsf{gmod}\text{-}R$, i.e. complexes M in $R\text{-}\mathsf{gmod}\text{-}R$ such that $M^i = 0$ for all but finitely many i. Given two bounded complexes, the morphism space is the bigraded total cohomology with composition the induced composition (3.3).

Note that $K^{\mathrm{b}}(R\text{-}\mathsf{gmod}\text{-}R)$ can be identified with the subcategory of $K^{\mathrm{b,bigr}}(R\text{-}\mathsf{gmod}\text{-}R)$ consisting of morphisms of degree $(0,0)$.

3.2. **Complexes of projective modules.** The category $R\text{-}\mathsf{gmod}\text{-}R$ is abelian. Let $\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R)$ be the full subcategory of projective objects. Then the natural functor

$$K^{\mathrm{b}}(R\text{-}\mathsf{gmod}\text{-}R) \xrightarrow{\sim} D^{\mathrm{b}}(R\text{-}\mathsf{gmod}\text{-}R) \tag{3.4}$$

is a triangulated equivalence.

Let $K^{\mathrm{b,bigr}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$ be the full subcategory of $K^{\mathrm{b,bigr}}(R\text{-}\mathsf{gmod}\text{-}R)$ consisting of complexes in $\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R)$. The subcategory of $K^{\mathrm{b,bigr}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$ of degree $(0,0)$ morphisms can be identified with $K^{\mathrm{b}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$.

Let B_1, B_2 be in $R\text{-}\mathsf{gmod}\text{-}R$, and fix projective resolutions $\epsilon_i : K_i \to B_i$, $i \in \{1,2\}$. We may view B_i as a complex supported in cohomological degree 0, and ϵ_i as chain maps in $\underline{\mathrm{Hom}}^{0,0}(K_i, B_i)$.

Consider a morphism $f^{\mathrm{bim}} : B_1 \to B_2(i)$ in $R\text{-}\mathsf{gmod}\text{-}R$, where $i \in \mathbb{Z}$. By the general theory of projective resolutions, there exists a chain map $\widetilde{f}$ in $\underline{\mathrm{Hom}}^{i,0}(K_1, K_2)$ lifting f^{bim}. That is, viewing f^{bim} as an element of $\underline{\mathrm{Hom}}^{i,0}(B_1, B_2)$, we have

$$\epsilon_2 \circ \widetilde{f} = f^{\mathrm{bim}} \circ \epsilon_1 \in \underline{\mathrm{Hom}}^{i,0}(K_1, B_2). \tag{3.5}$$

Moreover, all such $\widetilde{f}$ are homotopic, hence induce a well-defined morphism $f : K_1 \to K_2$ in $K^{\mathrm{b,bigr}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$ of degree $(i, 0)$.

3.3. **Monoidal structure on complexes.** Given two complexes M, M' in $R\text{-}\mathsf{gmod}\text{-}R$, define the tensor product complex $(M \otimes_R N, d_{M\otimes_R N})$ by

$$(M \otimes_R N)^i = \bigoplus_{i=p+q} (M^p \otimes N^q), \qquad d_{M\otimes_R N}(m \otimes n) = d_M(m) \otimes n + (-1)^m m \otimes d_N(n).$$

Let M, M', N, N' be complexes in $R\text{-}\mathsf{gmod}\text{-}R$, and let $f \in \underline{\mathrm{Hom}}(M, M')$ and $h \in \underline{\mathrm{Hom}}(N, N')$. Define the bigraded tensor product

$$- \otimes - : \underline{\mathrm{Hom}}(M, M') \otimes \underline{\mathrm{Hom}}(N, N') \to \underline{\mathrm{Hom}}(M \otimes_R N, M' \otimes_R N') \tag{3.6}$$

by

$$(f \otimes h)(m \otimes n) = (-1)^{|h||m|} f(m) \otimes h(n).$$

One checks directly that

$$d(f \otimes h) = d(f) \otimes h + (-1)^{|f|} f \otimes d(h). \tag{3.7}$$

It follows from (3.7) that the tensor product (3.6) induces a bigraded tensor product on total cohomology:

$$- \otimes - : \mathbb{H}\mathsf{om}(M, M') \otimes \mathbb{H}\mathsf{om}(N, N') \to \mathbb{H}\mathsf{om}(M \otimes_R N, M' \otimes_R N'). \tag{3.8}$$

Let L, L', M, M', N, N' be complexes in $R\text{-}\mathsf{gmod}\text{-}R$, and let $f \in \underline{\mathrm{Hom}}(M, N)$, $g \in \underline{\mathrm{Hom}}(L, M)$, $h \in \underline{\mathrm{Hom}}(M', N')$, $k \in \underline{\mathrm{Hom}}(L', M')$. One may directly check the following "super" interchange law:

$$(f \otimes h) \circ (g \otimes k) = (-1)^{|h||g|} (f \circ g) \otimes (h \circ k). \tag{3.9}$$

Passing to total cohomology, (3.9) continues to hold for the induced composition (3.3) and tensor (3.8).

Recall that a monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a category $\mathcal{C}$ together with a bifunctor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, identity object $\mathbb{1}$, and associator α and unitors λ, ρ satisfying certain axioms. Part of the bifunctoriality of $\otimes$ is the statement that morphisms in $\mathcal{C}$ satisfy the interchange law

$$(f \otimes h) \circ (g \otimes k) = (f \circ g) \otimes (h \circ k).$$

On $K^{\mathrm{b,bigr}}(R\text{-}\mathsf{gmod}\text{-}R)$, we have an operation $\otimes_R$ given on objects by $\otimes_R$ and on morphisms by the induced tensor (3.8). Let $\mathbb{1} = R$, viewed as a complex concentrated in cohomological degree 0. Natural isomorphisms α and λ, ρ can also be defined in the usual way. However, because of the sign in the super interchange law (3.9), these structures make $K^{\mathrm{b,bigr}}(R\text{-}\mathsf{gmod}\text{-}R)$ into not a monoidal category, but rather a structure that we call a *supermonoidal category with an additional grading*. That is, the morphisms spaces are bigraded, and morphisms satisfy the super interchange law for the first grading, as in (3.9). Moreover, there is an identity object $\mathbb{1}$ and associator α and unitors λ, ρ satisfy certain natural axioms we will not write down.

3.4. **Monoidal structure on complexes of projectives.** To give a similar super monoidal structure on $K^{\mathrm{b,bigr}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$, the monoidal identity R needs to be replaced by a projective resolution.

Define the complex $K_\varnothing$ by

$$K_\varnothing := \Lambda[\![1]\!] \otimes R^e \tag{3.10}$$

as a bigraded $\Bbbk$-module and with differential determined by

$$d(r \otimes (1 \otimes 1)) = 1 \otimes (r \otimes 1 - 1 \otimes r) \qquad \text{for all } r \in V^*[\![1]\!].$$

and the graded Leibniz rule (as usual with Koszul sign rule using the cohomological degree). Concretely,

$$d((r_1 \wedge \cdots \wedge r_k) \otimes f_1 \otimes f_2) = \sum_{i=1}^{k} (-1)^{i+1} (r_1 \wedge \cdots \wedge \widehat{r_i} \wedge \cdots r_k) \otimes (r_i f_1 \otimes f_2 - f_1 \otimes r_i f_2)$$

for $r_1, \ldots, r_k \in V^*[\![1]\!]$ and $f_1, f_2 \in R$. In fact, this makes $K_\varnothing$ into a dgga (a dga with an additional grading). This is just the Koszul resolution written in a basis-free way. In particular, the natural quotient map

$$\epsilon_{K_\varnothing} : K_\varnothing \xrightarrow{\sim} R$$

is a quasi-isomorphism.

For any complex M in $\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R)$, define degree $(0,0)$ chain maps

$$\widetilde{\lambda}_M : K_\varnothing \otimes_R M \xrightarrow{\epsilon_{K_\varnothing} \otimes \mathrm{id}_M} R \otimes_R M \xrightarrow{\text{mult}} M, \tag{3.11}$$

$$\widetilde{\rho}_M : M \otimes_R K_\varnothing \xrightarrow{\mathrm{id}_M \otimes \epsilon_{K_\varnothing}} M \otimes_R R \xrightarrow{\text{mult}} M, \tag{3.12}$$

where mult is multiplication, and let λ_M, ρ_M be the induced maps on total cohomology. Since $\epsilon_{K_\varnothing}$ is a quasi-isomorphism, the equivalence (3.4) shows that λ_M, ρ_M are isomorphisms in $K^{\mathrm{b}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$, hence also in $K^{\mathrm{b,bigr}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$. Since λ_M, ρ_M are functorial in M, they define natural isomorphisms $K_\varnothing \otimes_R - \cong \mathrm{id} \cong - \otimes_R K_\varnothing$ of functors in $K^{\mathrm{b,bigr}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$. Together with the obvious associator isomorphism, one checks that $K^{\mathrm{b,bigr}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$ becomes a supermonoidal category with an additional grading, with identity object $K_\varnothing$.

FINISH By fixing a projective resolution for each $M \in D^{\mathrm{b}}(R\text{-}\mathsf{gmod}\text{-}R)$, this gives a monoidal structure on $D^{\mathrm{b}}(R\text{-}\mathsf{gmod}\text{-}R)$ extending that on $R\text{-}\mathsf{gmod}\text{-}R$. The unitors for $R\text{-}\mathsf{gmod}\text{-}R$ are the natural isomorphisms

$$\lambda_B^{\mathrm{bim}} : R \otimes_R B \xrightarrow{\sim} B, \qquad \rho_B^{\mathrm{bim}} : B \otimes_R R \xrightarrow{\sim} B \tag{3.13}$$

given by multiplication. By the general consideration in §3.2, we obtain natural isomorphisms

$$\lambda_K : K_\varnothing \otimes_R K \to K, \qquad \rho_K : K \otimes_R K_\varnothing \to K \tag{3.14}$$

of degree $(0,0)$ for every projective resolution K.

3.5. **Bott–Samelson complexes.** For the rest of this section, return to the set-up of §2. Let $\mathrm{BSBim}(\mathfrak{h}, W)$ (resp. $\mathrm{SBim}(\mathfrak{h}, W)$) be the associated category of Bott–Samelson (resp. Soergel) bimodules. We assume further that $\mathrm{SBim}(\mathfrak{h}, W)$ satisfies the Soergel categorification theorem.

We have already defined a canonical resolution $\epsilon_{K_\varnothing} : K_\varnothing \to R$ in (3.10). Fix $s \in S$. By Demazure surjectivity, we may choose an element $\rho_s \in V^*$ satisfying $\langle \alpha_s^\vee, \rho_s \rangle = 1$. We will now define a resolution $\epsilon_{K_s} : K_s \to B_s$ that is canonical once we fix this choice of ρ_s. We need the following preliminary lemma.

Lemma 3.2. *The ring of s-invariants R^s is generated as a $\Bbbk$-algebra by $(V^*)^s$ and $\rho_s s(\rho_s)$.*

Proof. It is easy to see that $V = (V^*)^s \oplus \Bbbk\rho_s$, so

$$R = \mathrm{Sym}^\bullet(V^*) = \mathrm{Sym}^\bullet((V^*)^s)[\rho_s].$$

Given such a polynomial in ρ_s, we can repeatedly use the identity

$$\rho_s^2 = \rho_s(\rho_s + s(\rho_s)) - \rho_s s(\rho_s),$$

where $\rho_s + s(\rho_s) \in (V^*)^s$, to replace it with a polynomial in ρ_s of degree at most 1. Hence

$$R = \Bbbk\langle (V^*)^s, \rho_s s(\rho_s)\rangle + \rho_s \cdot \Bbbk\langle (V^*)^s, \rho_s s(\rho_s)\rangle.$$

Using this equality together with the containment $\Bbbk\langle (V^*)^s, \rho_s s(\rho_s)\rangle \subset R^s$, one quickly sees that $\Bbbk\langle (V^*)^s, \rho_s s(\rho_s)\rangle = R^s$. $\square$

It follows that the kernel of the structure map $R \otimes R(1) \twoheadrightarrow B_s$ is the ideal

$$(\{r \otimes 1 - 1 \otimes r\}_{r \in (V^*)^s}, \rho_s s(\rho_s) \otimes 1 - 1 \otimes \rho_s s(\rho_s)).$$

Moreover, if we choose a $\Bbbk$-basis $e_1, \ldots, e_{r-1}$ of $(V^*)^s$, then

$$(e_1 \otimes 1 - 1 \otimes e_1, \ldots, e_{r-1} \otimes 1 - 1 \otimes e_{r-1}, \rho_s s(\rho_s) \otimes 1 - 1 \otimes \rho_s s(\rho_s))$$

is a regular sequence in $R \otimes R$. Hence the associated Koszul complex is a resolution of B_s as a graded $R \otimes R$-module.

To express this more canonically, let

$$\xi_s = \rho_s s(\rho_s) \otimes 1 - 1 \otimes \rho_s s(\rho_s), \tag{3.15}$$

and define

$$K_s := \Lambda^\bullet\big((V^*)^s[\![1]\!] \oplus \Bbbk\xi_s[\![1]\!](-2)\big) \otimes R^e(1), \tag{3.16}$$

with differential determined by

$$d(r \otimes 1 \otimes 1) = 1 \otimes (r \otimes 1 - 1 \otimes r) \quad \text{ for all } r \in (V^*)^s[\![1]\!] \oplus \Bbbk\xi_s[\![1]\!](-2)$$

and the graded Leibniz rule.

Given an expression $(s_1, \ldots, s_m)$, the corresponding *Bott–Samelson complex* is the complex

$$\mathrm{BS}^{\mathrm{Ext}}(s_1, \ldots, s_m) := K_{s_1} \otimes_R \cdots \otimes_R K_{s_m}.$$

By convention, $\mathrm{BS}^{\mathrm{Ext}}(\varnothing) := K_\varnothing$.

Definition 3.3. The category $\mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ is the full subcategory of $K^{\mathrm{b,bigr}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$ consisting of complexes isomorphic to Bott–Samelson complexes. This is a full supermonoidal subcategory of $K^{\mathrm{b,bigr}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$.

3.6. **Ext-enhanced Soergel bimodules.** FINISH: explain how to recover $\mathrm{SBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ from $\mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$. This involves three steps:

(1) view $\mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ instead as a monoidal category with two compatible grading shifts. (this involves adding signs carefully to define the tensor product so that you get a genuine monoidal category)
(2) take additive envelope
(3) take Karoubi envelope

Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a supermonoidal category. Define a category $\mathcal{C}^{\mathrm{sh}}$ as follows. The objects of $\mathcal{C}^{\mathrm{sh}}$ are formal shifts $X[i]$, where X is an object of $\mathcal{C}$ and $i \in \mathbb{Z}$. Given two objects $X[n]$ and $X'[n']$, define

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{sh}}}(X[n], X'[n']) := \mathrm{Hom}_{\mathcal{C}}^{n'-n}(X, X),$$

and write $(f)_n^{n'}$ for the morphism corresponding to a degree $n' - n$ morphism f in $\mathcal{C}$. Composition in $\mathcal{C}^{\mathrm{sh}}$ is induced from $\mathcal{C}$ in the obvious way (involving no sign).

Next, we define $\otimes$ on $\mathcal{C}^{\mathrm{sh}}$. On objects,

$$X[n] \otimes Y[m] := X \otimes Y[n+m].$$

On morphisms,

$$(f)_a^b \otimes (g)_c^d := (-1)^{a|g|}(f \otimes g)_{a+c}^{b+d}$$

(coming from the Koszul sign rule $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$). One then checks that $\otimes$ satisfies the interchange law, without any sign.

3.7. **Some morphisms and relations in** $\mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$**.** In preparation for the diagrammatics, we will define some morphisms in $\mathrm{SBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ and find some relations between them.

3.7.1. *Ordinary generating morphisms.* Consider the graded bimodule homomorphisms

$$\begin{aligned}
\epsilon_s^{\mathrm{bim}} &: B_s \to R(1), f \otimes g \mapsto fg \\
\eta_s^{\mathrm{bim}} &: R \to B_s(1), f \mapsto f \cdot \Delta \\
m_s^{\mathrm{bim}} &: B_s \otimes_R B_s \to B_s(-1), f \otimes g \otimes h \mapsto \partial_s(g) \cdot f \otimes h \\
c_s^{\mathrm{bim}} &: B_s \to B_s \otimes_R B_s(-1), f \otimes g \mapsto f \otimes 1 \otimes g \\
m(f)^{\mathrm{bim}} &: R \to R(\deg f), g \mapsto fg, \text{ for } f \in R \text{ homogeneous.}
\end{aligned}$$

Here,

$$\Delta = \rho_s \otimes 1 - 1 \otimes s(\rho_s) \in B_s \tag{3.17}$$

is the canonical product-coproduct element coming from a Frobenius extension structure on $R^s \subset R$. The morphisms above are part of the generating morphisms of $\mathrm{BSBim}(\mathfrak{h}, W)$ in the diagrammatic presentations of Elias–Khovanov–Williamson, and in fact are all of them for $W = S_2$.

By the general consideration in §3.2, we obtain corresponding morphisms in $\mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$:

$$\begin{aligned} \epsilon_s &: K_s \to K_\varnothing(1), \\ \eta_s &: K_\varnothing \to K_s(1), \\ m_s &: K_s \otimes_R K_s \to K_s(-1), \\ c_s &: K_s \to K_s \otimes_R K_s(-1), \\ m_f &: K_\varnothing \to K_\varnothing(\deg f), \end{aligned}$$

Moreover, since the cohomological degree 0 part of $\mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ is monoidally equivalent to $\mathrm{BSBim}(\mathfrak{h}, W)$, these morphisms satisfy all the relations satisfied by corresponding bimodule homomorphisms.

It is of course possible to choose explicit chain maps representing each of the morphisms in $\mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, W)$ above, but we will only do this as necessary. For now, let us define a chain map

$$\widetilde{\epsilon}_s \in \underline{\mathrm{Hom}}^{1,0}(K_s, K_\varnothing) \tag{3.18}$$

representing ϵ_s.

First, recall that both $K_s(-1)$ and $K_\varnothing$ are dgga, and define a chain map

$$\widetilde{\epsilon}'_s \in \underline{\mathrm{Hom}}^{0,0}(K_s(-1), K_\varnothing)$$

by

$$\begin{aligned} \widetilde{\epsilon}'_s(r \otimes (1 \otimes 1)) &= r \otimes (1 \otimes 1), \\ \widetilde{\epsilon}'_s(\xi_s \otimes (1 \otimes 1)) &= \rho_s \otimes (s(\rho_s) \otimes 1) + s(\rho_s) \otimes (1 \otimes \rho_s), \end{aligned}$$

and extending multiplicatively. Then to check that $\widetilde{\epsilon}'_s$ is a chain map, it suffices to check it on these multiplicative generators. This is clear for $r \otimes (1 \otimes 1)$, and

$$d(\widetilde{\epsilon}'_s(\xi_s)) = \rho_s s(\rho_s) \otimes 1 - 1 \otimes \rho_s s(\rho_s) = \widetilde{\epsilon}'_s(d(\xi_s)).$$

Now, the chain map $\widetilde{\epsilon}_s$ is defined to be the image of the chain map $\widetilde{\epsilon}'_s$ under the obvious identification (involving no sign)

$$\underline{\mathrm{Hom}}^{0,0}(K_s(-1), K_\varnothing) \xrightarrow{\sim} \underline{\mathrm{Hom}}^{1,0}(K_s, K_\varnothing).$$

Moreover, it is clear from the definition that $\widetilde{\epsilon}_s$ represents ϵ_s.

3.7.2. *Endomorphisms of $K_\varnothing$.* The derivation (2.4) induces a chain map $\Lambda^\vee \otimes K_\varnothing \to K_\varnothing$. Let

$$\widetilde{\iota} : \Lambda^\vee \to \underline{\mathrm{End}}(K_\varnothing)$$

be the associated bigraded $\Bbbk$-algebra map, and let ι_x be the class of the chain map $\widetilde{\iota}_x = x \acute{\frown} (-)$.

Lemma 3.4. *The bigraded $\Bbbk$-algebra map*

$$R_{\mathrm{Ext}} \to \mathbb{E}\mathsf{nd}(K_\varnothing)$$

determined by

$$f \otimes x \mapsto m_f \circ \iota_x = \iota_x \circ m_f$$

is an isomorphism.

Proof. FINISH $\square$

This makes every morphism space in $K^{\mathrm{b,bigr}}(\mathsf{Proj}(R\text{-}\mathsf{gmod}\text{-}R))$ into a bigraded $(R \otimes \Lambda^\vee)$-bimodule. Concretely, given $f \otimes x \in R \otimes \Lambda^\vee$ and $g : K_1 \to K_2$, we have

$$(f \otimes x) \cdot g = \rho_{K_2} \circ ((m_f \circ \iota_x) \otimes g) \circ \rho_{K_1}^{-1}, \tag{3.19}$$

$$g \cdot (f \otimes x) = \lambda_{K_2} \circ (g \otimes (m_f \circ \iota_x)) \circ \lambda_{K_1}^{-1}. \tag{3.20}$$

3.7.3. *Monoidal structure.* Recall that the exterior algebra Λ is naturally a Hopf algebra. In particular, there is a multiplicative comultiplication $\Delta : \Lambda \to \Lambda \otimes \Lambda$ determined by $\Delta(r) = r \otimes 1 + 1 \otimes r$ for $r \in V^* \subset \Lambda$. Extending R^e-linearly, we obtain a multiplicative map $\Delta_{K_\varnothing} \in \underline{\mathrm{Hom}}^{0,0}(K_\varnothing, K_\varnothing \otimes K_\varnothing)$.

We claim that $\Delta_{K_\varnothing}$ is a chain map:

$$\Delta_{K_\varnothing} \circ d_{K_\varnothing} = d_{K_\varnothing \otimes K_\varnothing} \circ \Delta_{K_\varnothing}. \tag{3.21}$$

Indeed, since $K_\varnothing$ and $K_\varnothing \otimes K_\varnothing$ are both dgga and $\Delta_{K_\varnothing}$ is multiplicative, it suffices to check (3.21) on $r \otimes (1 \otimes 1)$ for $r \in V^* \subset \Lambda$, where it is easy.

Next, we define a chain map

$$\widetilde{\tau}_s \in \underline{\mathrm{Hom}}^{0,0}(K_s, K_\varnothing \otimes_R K_s) \tag{3.22}$$

representing the morphism $\lambda_{K_s}^{-1}$, where λ is the left unitor (see (3.14)). To do this, note that $K_s(-1)$ is also a dgga, so $K_\varnothing \otimes_R K_s(-1)$ is also a dgga, and first define an element

$$\widetilde{\tau}'_s \in \underline{\mathrm{Hom}}^{0,0}(K_s(-1), K_\varnothing \otimes_R K_s(-1))$$

by

$$\begin{aligned} \widetilde{\tau}'_s(r \otimes (1 \otimes 1)) &= [r \otimes (1 \otimes 1)] \otimes [1] + [1] \otimes [r \otimes (1 \otimes 1)] \quad \text{ for } r \in (V^*)^s \subset \Lambda, \\ \widetilde{\tau}'_s(\xi_s \otimes (1 \otimes 1)) &= [\rho_s \otimes (s(\rho_s) \otimes 1) + s(\rho_s) \otimes (1 \otimes \rho_s)] \otimes [1] + [1] \otimes [\xi_s \otimes (1 \otimes 1)] \end{aligned}$$

and extending multiplicatively. (This is possible because the comultiplication Δ on Λ restricts to one on Λ^s.) As with $\Delta_{K_\varnothing}$, we can show by checking on these multiplicative generators that $\widetilde{\tau}'_s$ is a chain map. The only new computation is for ξ_s:

$$\begin{aligned} d(\widetilde{\tau}'_s(\xi_s)) &= [\rho_s s(\rho_s) \otimes 1 - 1 \otimes \rho_s s(\rho_s)] \otimes [1] + [1] \otimes [\rho_s s(\rho_s) \otimes 1 - 1 \otimes \rho_s s(\rho_s)] \\ &= [\rho_s s(\rho_s) \otimes 1] \otimes [1] - [1] \otimes [1 \otimes \rho_s s(\rho_s)] \\ &= \widetilde{\tau}'_s(\rho_s s(\rho_s) \otimes 1 - 1 \otimes \rho_s s(\rho_s)) = \widetilde{\tau}'_s(d(\xi_s)). \end{aligned}$$

Thus $\widetilde{\tau}'_s$ is a chain map. Now, we let $\widetilde{\tau}_s$ be the image of $\widetilde{\tau}'_s$ under the obvious identification (involving no sign)

$$\underline{\mathrm{Hom}}^{0,0}(K_s(-1), K_\varnothing \otimes_R K_s(-1)) \xrightarrow{\sim} \underline{\mathrm{Hom}}^{0,0}(K_s, K_\varnothing \otimes_R K_s).$$

Clearly

$$(\epsilon_{K_\varnothing} \otimes \epsilon_{K_s}) \circ \widetilde{\tau}_s = (\lambda_{B_s}^{\mathrm{bim}})^{-1} \circ \epsilon_{K_s},$$

so $\widetilde{\tau}_s$ represents $\lambda_{K_s}^{-1}$, as claimed.

One may similarly define a chain map

$$\widetilde{\sigma}_s \in \underline{\mathrm{Hom}}^{0,0}(K_s, K_s \otimes_R K_\varnothing) \tag{3.23}$$

representing $\rho_{K_s}^{-1}$ as the image of $\widetilde{\sigma}'_s \in \underline{\mathrm{Hom}}^{0,0}(K_s(-1), K_s(-1) \otimes_R K_\varnothing)$ defined by

$$\begin{aligned} \widetilde{\sigma}'_s(r \otimes (1 \otimes 1)) &= [r \otimes (1 \otimes 1)] \otimes [1] + [1] \otimes [r \otimes (1 \otimes 1)] \quad \text{ for } r \in (V^*)^s \subset \Lambda, \\ \widetilde{\sigma}'_s(\xi_s \otimes (1 \otimes 1)) &= [\xi_s \otimes (1 \otimes 1)] \otimes [1] + [1] \otimes [\rho_s \otimes (s(\rho_s) \otimes 1) + s(\rho_s) \otimes (1 \otimes \rho_s)], \end{aligned}$$

and extending multiplicatively.

3.7.4. *The morphism ϕ_s.* Extend the canonical isomorphism (2.3) to

$$((V^*)^s \oplus \Bbbk\xi_s)^* = (V/\Bbbk\alpha_s^\vee) \oplus \Bbbk\xi_s^\vee \tag{3.24}$$

by letting any element of $V/\Bbbk\alpha_s^\vee$ act as 0 on ξ_s, and by setting $\xi_s^\vee((V^*)^s) = 0$ and $\xi_s^\vee(\xi_s) = 1$. It follows that

$$\begin{aligned}\underline{\mathrm{Hom}}(K_s, B_s) = \underline{\mathrm{Hom}}\left(\Lambda^\bullet\big((V^*)^s[\![1]\!] \oplus \Bbbk\xi_s(-2)[\![1]\!]\big) \otimes R^e(1), B_s\right)\\ \cong B_s(-1) \otimes \Lambda^\bullet\big((V/\Bbbk\alpha_s^\vee)[\![-1]\!] \oplus \Bbbk\xi_s^\vee(2)[\![-1]\!]\big)\end{aligned}$$

as bigraded $\Bbbk$-modules. Moreover, $d_{\underline{\mathrm{Hom}}(K_s,B_s)} = 0$, so the total cohomology is the same. We deduce that

$$\dim_\Bbbk \mathbb{H}\mathsf{om}^{-2,1}(K_s, B_s) = 1.$$

Since

$$\epsilon_{K_s} \circ - : \mathrm{Hom}_{K^\mathrm{b}}(K_s, K_s[i](j)) \to \mathrm{Hom}_{K^\mathrm{b}}(K_s, B_s[i](j)) \tag{3.25}$$

is an isomorphism for any $(i, j) \in \mathbb{Z}^2$, it follows that

$$\dim \mathbb{H}\mathsf{om}^{-2,1}(K_s, K_s) = 1$$

as well.

For degree reasons, any chain map $f \in \underline{\mathrm{Hom}}^{-2,1}(K_s, B_s)$ sends $f((K_s)^i) = 0$ if $i \neq -1$, $f((V^*)^s \otimes R^e) = 0$, and $f(\xi_s) = a(1 \otimes 1)$ for some $a \in \Bbbk$.

Since K_s is a Koszul resolution, there is a chain map

$$\widetilde{\phi_s} = -\xi_s^\vee \frown (-) \in \underline{\mathrm{Hom}}^{-2,1}(K_s, K_s). \tag{3.26}$$

Let

$$\phi_s \in \mathbb{H}\mathsf{om}^{-2,1}(K_s, K_s) \tag{3.27}$$

be the induced map on total cohomology. Since $(\epsilon_{K_s} \circ \widetilde{\phi_s})(\xi_s) = -1 \otimes 1$, the following lemma is immediate from the discussion above.

Lemma 3.5. *Let $f \in \mathbb{H}\mathsf{om}^{-2,1}(K_s, K_s)$. Then $f = -a\phi_s$, where $a \in \Bbbk$ is determined by $(\epsilon_{K_s} \circ \widetilde{f})(\xi_s) = a(1 \otimes 1)$ for any chain map $\widetilde{f}$ representing f.*

Remark 3.6. The minus sign in the definition of ϕ_s is chosen so that the Hochschild barbell relation (4.4) involves $\alpha_s^\vee$ rather than $-\alpha_s^\vee$.

3.7.5. *The morphism η_s^{Ext}.* Since K_s is also a Koszul resolution, we may define chain maps in $\underline{\mathrm{End}}(K_s)$ by contraction, as in §3.7.2. Define the chain map

$$\widetilde{\eta}_s^{\mathrm{Ext}} = \alpha_s^\vee \mathbin{\acute{\frown}} (-) \in \underline{\mathrm{Hom}}^{-1,1}(K_\varnothing, K_s). \tag{3.28}$$

Here, we use the isomorphism (3.24), so $\widetilde{\eta}_s^{\mathrm{Ext}}$ kills ξ_s. Let

$$\eta_s^{\mathrm{Ext}} \in \mathbb{H}\mathsf{om}^{-1,1}(K_\varnothing, K_s) \tag{3.29}$$

be the induced morphism.

Lemma 3.7. *We have*

$$m_s \circ (\eta_s^{\mathrm{Ext}} \otimes \mathrm{id}_{K_s}) \circ \lambda_{K_s}^{-1} = \phi_s = m_s \circ (\mathrm{id}_{K_s} \otimes \eta_s^{\mathrm{Ext}}) \circ \rho_{K_s}^{-1}. \tag{3.30}$$

Proof. We prove the first equality using Lemma 3.5 and the chain map $\widetilde{\tau}$ defined in (3.22); the second equality is similar, but using the chain map $\widetilde{\sigma}$ defined in (3.23).

Arbitrary choose a chain map $\widetilde{m}_s$ representing m_s, and consider the chain map

$$\widetilde{m}_s \circ (\widetilde{\eta}_s^{\mathrm{Ext}} \otimes \mathrm{id}_{K_s}) \circ \widetilde{\tau}_s \in \underline{\mathrm{Hom}}^{-2,1}(K_s, K_s)$$

representing $m_s \circ (\eta_s^{\mathrm{Ext}} \otimes \mathrm{id}_{K_s}) \circ \lambda_{K_s}^{-1}$. By (3.5), we have

$$\epsilon_{K_s} \circ \widetilde{m}_s \circ (\widetilde{\eta}_s^{\mathrm{Ext}} \otimes \mathrm{id}_{K_s}) \circ \widetilde{\tau}_s = m_s^{\mathrm{bim}} \circ (\epsilon_{K_s} \otimes \epsilon_{K_s}) \circ (\widetilde{\eta}_s^{\mathrm{Ext}} \otimes \mathrm{id}_{K_s}) \circ \widetilde{\tau}_s.$$

We can now explicitly compute the effect of the last expression on ξ_s, step by step:

$$\begin{aligned}\xi_s &\xmapsto{\widetilde{\tau}_s} [\rho \otimes (s(\rho) \otimes 1) + s(\rho) \otimes (1 \otimes \rho)] \otimes [1] + [1] \otimes [\xi_s \otimes (1 \otimes 1)] \\ &\xmapsto{\widetilde{\eta}_s^{\mathrm{Ext}} \otimes \mathrm{id}_{K_s}} [s(\rho) \otimes 1 - 1 \otimes \rho] \otimes [1] \xmapsto{\epsilon_{K_s} \otimes \epsilon_{K_s}} s(\rho) \otimes 1 \otimes 1 - 1 \otimes \rho \otimes 1 \xmapsto{m_s^{\mathrm{bim}}} -1.\end{aligned}$$

Hence the first equality follows from Lemma 3.5. □

Lemma 3.8. *We have*

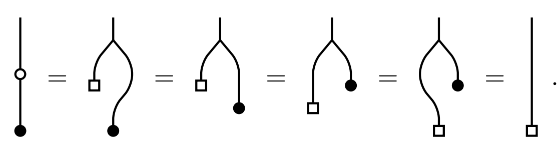
$$\eta_s^{\mathrm{Ext}} = \phi_s \circ \eta_s.$$

Proof. This follows from the following chain of equalities:

$$\begin{aligned}\phi_s \circ \eta_s &= m_s \circ (\eta_s^{\mathrm{Ext}} \otimes \mathrm{id}_{K_s}) \circ \lambda_{K_s}^{-1} \circ \eta_s \\ &= m_s \circ (\eta_s^{\mathrm{Ext}} \otimes \mathrm{id}_{K_s}) \circ (\mathrm{id}_{K_\varnothing} \otimes \eta_s) \circ \lambda_{K_\varnothing}^{-1} \\ &= m_s \circ (\mathrm{id}_{K_s} \otimes \eta_s) \circ (\eta_s^{\mathrm{Ext}} \otimes \mathrm{id}_{K_\varnothing}) \circ \lambda_{K_\varnothing}^{-1} \\ &= m_s \circ (\mathrm{id}_{K_s} \otimes \eta_s) \circ \rho_{K_s}^{-1} \circ \rho_{K_s} \circ (\eta_s^{\mathrm{Ext}} \otimes \mathrm{id}_{K_\varnothing}) \circ \lambda_{K_\varnothing}^{-1} \\ &= m_s \circ (\mathrm{id}_{K_s} \otimes \eta_s) \circ \rho_{K_s}^{-1} \circ \eta_s^{\mathrm{Ext}} \circ \rho_{K_\varnothing} \circ \lambda_{K_\varnothing}^{-1} \\ &= \eta_s^{\mathrm{Ext}}.\end{aligned}$$

The first equality follows from Lemma 3.7. The third equality follows from the interchange law. The second and fifth equalities follow from the naturality of λ^{-1} and ρ, respectively. The last equality follows from the corresponding relations in R-gmod-R. □

Remark 3.9. The proof of Lemma 3.8 above is best understood diagrammatically:



3.7.6. *Exterior forcing relation.*

Lemma 3.10. *For any $x \in V[\![-1]\!] \subset \Lambda^\vee$, we have*

$$\mathrm{id}_{K_s} \cdot x - s(x) \cdot \mathrm{id}_{K_s} = \alpha_s(x)(\eta_s^{\mathrm{Ext}} \circ \epsilon_s) \tag{3.31}$$

in $\mathbb{E}\mathsf{nd}^{0,1}(K_s)$.

Proof. Because of the isomorphism (3.25), it suffices to show (3.31) after post-composing with ϵ_{K_s}, i.e. show that

$$\epsilon_{K_s} \circ (\mathrm{id}_{K_s} \cdot x - s(x) \cdot \mathrm{id}_{K_s}) = \alpha_s(x)(\epsilon_{K_s} \circ \eta_s^{\mathrm{Ext}} \circ \epsilon_s) \tag{3.32}$$

in $\mathbb{H}\mathsf{om}^{0,1}(K_s, B_s)$. By the definition of the $(R \otimes \Lambda^\vee)$-bimodule structure (3.19), the left hand side of (3.32) can be represented by the chain map

$$\epsilon_{K_s} \circ (\widetilde{\rho}_{K_s} \circ (\mathrm{id}_{K_s} \otimes \widetilde{\iota}_x) \circ \widetilde{\sigma}_s - \widetilde{\lambda}_{K_s} \circ (\widetilde{\iota}_{s(x)} \otimes \mathrm{id}_{K_s}) \circ \widetilde{\tau}_s). \tag{3.33}$$

The right hand side of (3.32) can be represented by the chain map

$$\alpha_s(x)(\epsilon_{K_s} \circ \widetilde{\eta}_s^{\mathrm{Ext}} \circ \widetilde{\epsilon}_s). \tag{3.34}$$

In fact, we will show by an explicit calculation that (3.33) and (3.34) are already equal as chain maps. Since any element in $\underline{\mathrm{Hom}}^{0,1}(K_s, B_s)$ is automatically zero outside $(K_s)^{-1}$, we only need to understand what happens to $(V^*)^s f \otimes (1 \otimes 1)$ and to $\xi_s \otimes (1 \otimes 1)$.

From the explicit description of the chain maps $\widetilde{\epsilon}_s$ (3.18) and $\widetilde{\eta}_s^{\mathrm{Ext}}$ (3.28), we see that (3.34) kills $(V^*)^s \otimes (1 \otimes 1)$ and sends ξ_s to $\alpha_s(x)(s(\rho_s) \otimes 1 - 1 \otimes \rho_s)$.

For (3.33), we compute using the explicit descriptions of $\widetilde{\tau}_s$ (3.22) and $\widetilde{\sigma}_s$ (3.23). For $r \in (V^*)^s$, we have

$$r \xmapsto{\widetilde{\tau}_s} [r] \otimes [1] + [1] \otimes [r] \xmapsto{\widetilde{\iota}_{s(x)} \otimes \mathrm{id}_{K_s}} \langle s(x), r\rangle \xmapsto{\widetilde{\lambda}_{K_s}} \langle s(x), r\rangle,$$
$$r \xmapsto{\widetilde{\sigma}_s} [r] \otimes [1] + [1] \otimes [r] \xmapsto{\mathrm{id}_{K_s} \otimes \widetilde{\iota}_x} \langle x, r\rangle[1] \otimes [1] \xmapsto{\widetilde{\rho}_{K_s}} \langle x, r\rangle,$$

and $\langle s(x), r\rangle = \langle x, s(r)\rangle = \langle x, r\rangle$. Moreover,

$$\begin{aligned}\xi_s &\xmapsto{\widetilde{\tau}_s} [\rho_s \otimes (s(\rho_s) \otimes 1) + s(\rho_s) \otimes (1 \otimes \rho_s)] \otimes [1] + [1] \otimes [\xi_s] \\ &\xmapsto{\widetilde{\iota}_{s(x)} \otimes \mathrm{id}_{K_s}} [\langle x, s(\rho_s)\rangle s(\rho_s) \otimes 1 + \langle x, \rho_s\rangle 1 \otimes \rho_s] \otimes [1] \\ &\xmapsto{\widetilde{\lambda}_{K_s}} \langle x, s(\rho_s)\rangle s(\rho_s) \otimes 1 + \langle x, \rho_s\rangle 1 \otimes \rho_s\end{aligned}$$

and

$$\begin{aligned}\xi_s &\xmapsto{\widetilde{\sigma}_s} [\rho_s \otimes (s(\rho_s) \otimes 1) + s(\rho_s) \otimes (1 \otimes \rho_s)] \otimes [1] + [1] \otimes [\xi_s] \\ &\xmapsto{\mathrm{id}_{K_s} \otimes \widetilde{\iota}_x} [\langle x, \rho_s\rangle s(\rho_s) \otimes 1 + \langle x, s(\rho_s)\rangle 1 \otimes \rho_s] \otimes [1] \\ &\xmapsto{\widetilde{\rho}_{K_s}} \langle x, \rho_s\rangle s(\rho_s) \otimes 1 + \langle x, s(\rho_s)\rangle 1 \otimes \rho_s.\end{aligned}$$

It follows that (3.33) also kills $(V^*)^s \otimes (1 \otimes 1)$ and sends ξ_s to

$$\langle x, \rho_s - s(\rho_s)\rangle s(\rho_s) \otimes 1 + \langle x, s(\rho_s) - \rho_s\rangle 1 \otimes \rho_s = \alpha_s(x)(s(\rho_s) \otimes 1 - 1 \otimes \rho_s).$$

This completes the proof. □

Remark 3.11. For the GL_2-realization $\mathfrak{h}$, many of the relations between morphisms in $\mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, S_2)$ are already in [GH, §3.5]. The correspondence between the notation for morphisms in this paper and that of [GH] is as follows:

$$\eta_s = b^*, \quad \epsilon_s = b, \quad \eta_s^{\mathrm{Ext}} = \omega^*, \quad \epsilon_s^{\mathrm{Ext}} = \omega, \quad \phi_s = -\iota_{\varphi_2}.$$

4. Diagrammatics

4.1. Definition of the digrammatic category.

Definition 4.1. The *Ext-enhanced Elias–Williamson diagrammatic category* associated to the realization $\mathfrak{h}$ of $W = W(A_1)$, denoted by

$$\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W),$$

is the strict $\Bbbk$-linear supermonoidal category with an additional grading defined by the diagrammatic presentation below.

More precisely, morphisms spaces of $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ are bigraded $\Bbbk$-modules. We write $\deg f = (m, n)$ to mean that a morphism f is homogeneous of bidegree (m, n), or *cohomological degree* m and *Soergel degree* n. If a morphism f is homogeneous, we also denote its cohomological degree by $|f|$. The category $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ is supermonoidal for the cohomological grading. That is, there is a tensor product operation $\otimes$ and an identity object satisfying the axioms of a strict monoidal category, except that $\otimes$ is not bifunctorial on morphisms, but rather satisfy the *super exchange law* for the cohomological grading:

$$(h \otimes k) \circ (f \otimes g) = (-1)^{|k||f|}(h \circ f) \otimes (k \circ g)$$

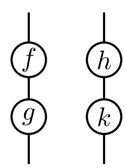
for homogeneous h, k.

4.1.1. *Objects.* The objects of $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ are the same as those of the ordinary Elias–Williason diagrammatic category $\mathscr{D}(\mathfrak{h}, W)$: they are indexed by expressions, i.e. words in $S = \{s\}$, and the object corresponding $\underline{w}$ is denoted by $B_{\underline{w}}$. In other words, the objects of $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ are

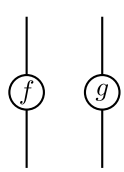
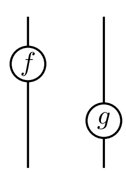
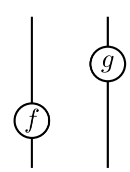
$$B_\varnothing, \quad B_s, \quad B_{(s,s)}, \quad B_{(s,s,s)}, \quad \cdots .$$

4.1.2. *Morphisms.* The Elias–Khovanov diagrammatic category is a $\Bbbk$-linear strict monoidal category described by the well-known string diagrams, where each diagram represents a morphism from its bottom boundary to its top boundary, and the monoidal structure and composition correspond, respectively, to horizontal and vertical stacking of diagrams.

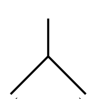
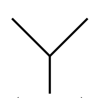
The string diagrammatics can be adapted to our setting of a supermonoidal category with an additional grading as follows. A diagram is to be interpreted by composing horizontally and then vertically, so that


$$= (f \otimes h) \circ (g \otimes k).$$

Then the super interchange law follows from the following more basic interchange law:


$$=$$

$$= (-1)^{|f||g|}$$


As in the Elias–Khovanov category, the morphisms in $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ are specified by a set of generating morphisms. Each generating morphism in $\mathscr{D}(\mathfrak{h}, W)$ of degree m is also a generating morphism in $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ of bidegree $(m, 0)$.

generator					$\boxed{f}$
bidegree	$(1, 0)$	$(1, 0)$	$(-1, 0)$	$(-1, 0)$	$(\deg f, 0)$

Here, f is a homogeneous element in R. In addition, $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ has the following "Hochschild generators":

generator		$\boxed{x}$
bidegree	$(-2, 1)$	$\deg x$
name	(bivalent) Hochschild dot	exterior box

Here, x is a homogeneous element in $\Lambda^\vee$, and $\deg x$ denotes its bidegree. For example, $x \in V \subset \Lambda^\vee$ has bidegree $(0, 1)$.

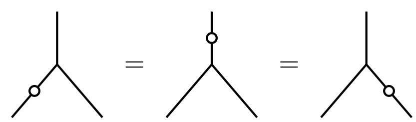
In other words, a general morphism $B_{\underline{w}} \to B_{\underline{v}}$ in $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ is represented by a $\Bbbk$-linear combination of planar diagrams, where each diagram has bottom boundary $\underline{w}$, top boundary $\underline{v}$, and is made up of local pieces given by the generating morphisms above.

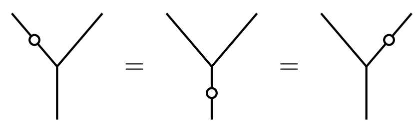
We also define the following "univalent Hochschild dots" as shorthands:

(4.1) 

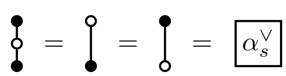
4.1.3. *Relations.* The morphisms of $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ satisfy the defining relations of $\mathscr{D}(\mathfrak{h}, W)$, plus the following additional relations:

(1) *Hochschild dot slides past trivalent vertices:*

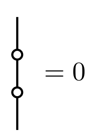
(4.2) 

(4.3) 

(2) *Hochschild barbell relation:*

(4.4) 

(3) *Hochschild dot annihilation:*

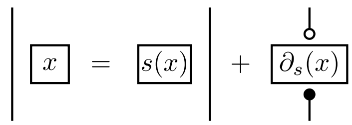
(4.5) 

(4) *Exterior boxes add and multiply:*

$$\boxed{x} + \boxed{y} = \boxed{x+y}\,, \qquad \begin{matrix}\boxed{x}\\ \\ \boxed{y}\end{matrix} = \boxed{x \wedge y} \tag{4.6}$$

for $x, y \in \Lambda^\vee$.

(5) *Exterior forcing relation:*

(4.7)  for $x \in \Lambda^\vee$ homogeneous,

where $\partial_s : \Lambda^\vee \to \Lambda_s^\vee = (\Lambda^\vee)/\Bbbk\alpha_s^\vee$ is the exterior Demazure operator defined in §2.3.

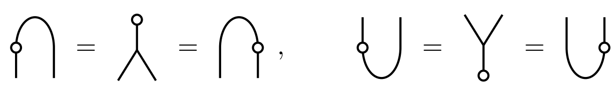
Remark 4.2. Although $\partial_s(x)$ is only defined modulo $\Bbbk\alpha_s^\vee$, the final term in (4.7) makes sense because of the relation (4.14), which is derived without using (4.7).

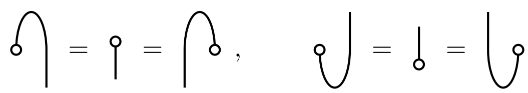
This concludes the list of the additional relations and the definition of $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$. Note that all relations are homogeneous for the bigrading, so the morphism spaces in $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ are bigraded.

4.2. **Further relations.**

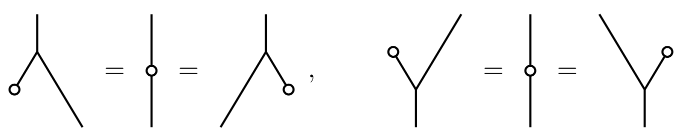
Lemma 4.3. *The following relations hold in* $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$.

(1) Hochschild dot is cyclic:

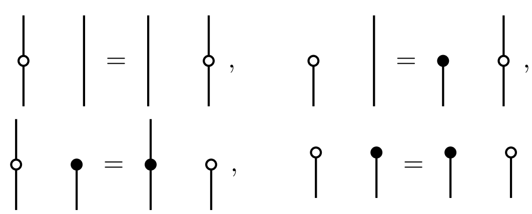
(4.8) 

(4.9) 

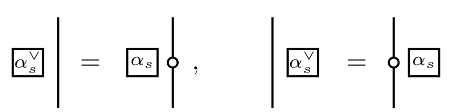
(2) Hochschild dot "pops out":

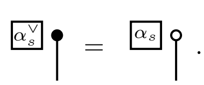
(4.10) 

(3) Hochschild dot jumps:

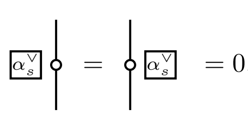
(4.11) 

(4) Ordinary vs. Hochschild dot*:*

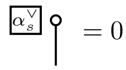
(4.12) 

(4.13) 

(5) Simple coroot kills Hochschild dot:

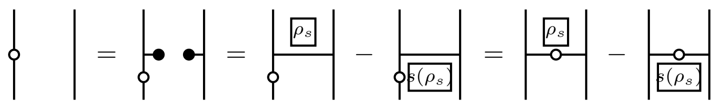
(4.14) 

(4.15)



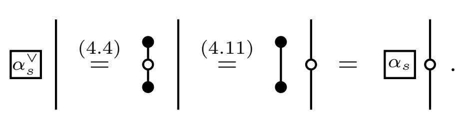
Proof. The equations in (4.8) and (4.9) are obtained by adding an ordinary dot morphism to (4.2) and (4.3) and simplifying using the Elias–Khovanov one-color relations.

The first equation in (4.11) follows from the calculation



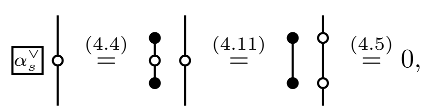
by noting that the last expression is fixed by reflection about a vertical axis. The other equations in (4.11) are obtained by dotting the first equation.

The first equation in (4.12) follows from the computation



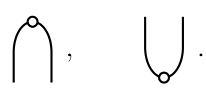
The second equation is similar. Equation (4.13) is obtained by dotting (4.12).

Finally, the first equation in (4.14) follows from

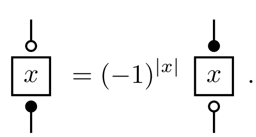


and the second equation is similar. Equation (4.15) is obtained by dotting (4.14). ☐

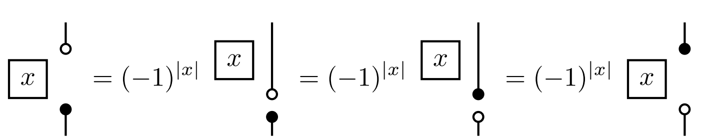
By cyclicity, diagrams in $\mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, W)$ up to isotopy unambigiuously represents a morphism. For example, the morphisms in (4.8) can be drawn as



For any homogeneous $x \in \Lambda$, we have

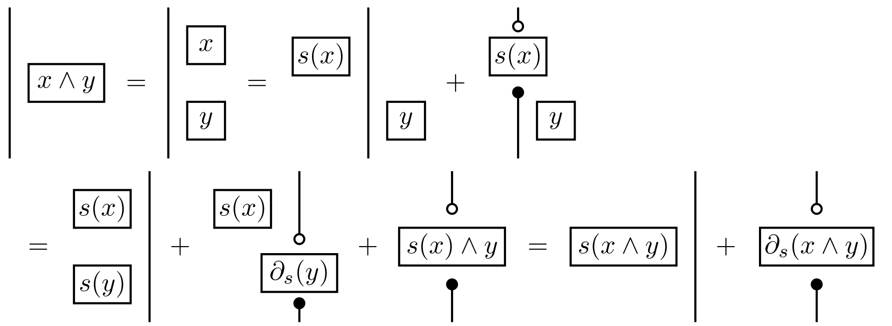


Indeed,



We can therefore

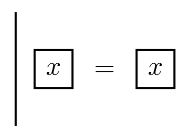
Remark 4.4. If the exterior forcing relation (4.7) is known for exterior boxes labeled x and y, then the exterior forcing relation for $x \wedge y$ can be deduced from the remaining relations. Indeed,



where the last equality uses the twisted graded Leibniz rule (2.9) for ∂_s.

Remark 4.5. If 2 is invertible in $\Bbbk$, then the exterior forcing relation (4.7) can be replaced by the weaker relation

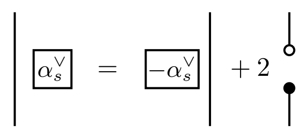
(4.16)



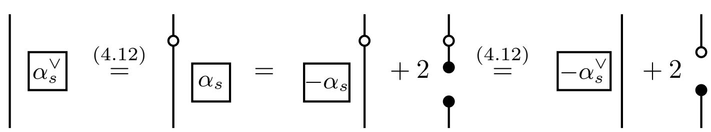
for $x \in (\Lambda^\vee)^s$.

First, we observe that the special case of the exterior forcing relation

(4.17)



can be derived from the other relations:



Now, if 2 is invertible in $\Bbbk$, then any $x \in \Lambda^\vee$ can be decomposed as

$$x = \frac{x + s(x)}{2} + \frac{1}{2}\alpha_s^\vee \wedge \partial_s(x),$$

where $\frac{x+s(x)}{2}, \partial_s(x) \in (\Lambda^\vee)^s$. Using this decomposition, (4.7) can be derived from (4.16) and (4.17).

Even if 2 is not invertible in $\Bbbk$, one still has the (unique) decomposition $x = y + \rho_s^\vee \wedge z$ for $y, z \in (\Lambda^\vee)^s$. However, we still need to know how to slide $\rho_s^\vee$.

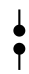
4.3. **Computation of morphism spaces.** The following morphism spaces were computed in [EK].

Lemma 4.6. *We have*

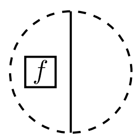
$$\operatorname{Hom}_{\mathscr{D}}(B_\varnothing, B_\varnothing) = R \cdot \mathrm{id}_{B_\varnothing},$$

$$\operatorname{Hom}_{\mathscr{D}}(B_s, B_\varnothing) = R \cdot$$  ,

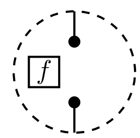
$$\operatorname{Hom}_{\mathscr{D}}(B_\varnothing, B_s) = R \cdot$$  ,

$$\operatorname{Hom}_{\mathscr{D}}(B_s, B_s) = R \cdot$$ $+ R \cdot$ 

For instance, the last equality says that every ordinary Elias–Khovanov diagram with bottom boundary s and top boundary s can be reduced to a $\Bbbk$-linear combination of diagrams of the forms



or

 ,

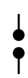
where f is a homogeneous element of R. We use Lemma 4.6 to prove the following results for morphism spaces in $\mathscr{D}^{\mathrm{Ext}}$.

Proposition 4.7. *We have*

$$\operatorname{Hom}_{\mathscr{D}^{\mathrm{Ext}}}(B_\varnothing, B_\varnothing) = R_{\mathrm{Ext}} = R \otimes \Lambda^\vee, \tag{4.18}$$

(4.19) $\operatorname{Hom}_{\mathscr{D}^{\mathrm{Ext}}}(B_s, B_\varnothing) = R_{\mathrm{Ext}} \cdot$  $+ R_{\mathrm{Ext}} \cdot$  ,

(4.20) $\operatorname{Hom}_{\mathscr{D}^{\mathrm{Ext}}}(B_\varnothing, B_s) = R_{\mathrm{Ext}} \cdot$  $+ R_{\mathrm{Ext}} \cdot$  ,

(4.21) $\operatorname{Hom}_{\mathscr{D}^{\mathrm{Ext}}}(B_s, B_s) \;=\; R_{\mathrm{Ext}} \cdot$ $+ R_{\mathrm{Ext}} \cdot$  $+ R_{\mathrm{Ext}} \cdot$  $+ R_{\mathrm{Ext}} \cdot$  ,

Moreover, for each diagram above containing a Hochschild dot, the coefficient $f \otimes x \in R \otimes \Lambda^\vee$ can be chosen so that f is not divisible by α_s and x is not divisible by $\alpha_s^\vee$.

For example, (4.18) says that every morphism $B_\varnothing \to B_\varnothing$ in $\mathscr{D}^{\mathrm{Ext}}$ can be reduced to a finite sum of diagrams of the form

$$\boxed{x} \quad \boxed{f}$$

for some $x \in \Lambda^\vee$ homogeneous and $f \in R$ homogeneous. However, it does not say that $\operatorname{End}_{\mathscr{D}^{\mathrm{Ext}}}(B_\varnothing) \cong R \otimes \Lambda^\vee$ since a priori we may have $\mathrm{id}_{B_\varnothing} = 0$.

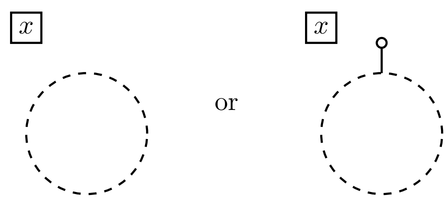
In the proof below, given a diagram, the connected components of the complement of the underlying graph in $\mathbb{R} \times [0,1]$ are called its *regions*. The *vertices* of the graph are its points on the boundaries $\mathbb{R} \times 0, 1$ and the ordinary (non-Hochschild) univalent and trivalent vertices. An *edge* of the graph is a connected component of the complement of the vertices.

Proof. Consider the following procedure on a diagram containing at least one edge:

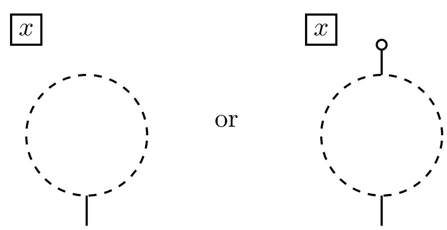
(1) Use the exterior forcing relation to move every exterior box to the far left region, at the cost of replacing a diagram with a $\Bbbk$-linear combination of diagrams. (The subsequent steps of this procedure are to be applied to each diagram.)
(2) Choose a "highest" edge, i.e. one that borders the same region as the top boundary, and use (4.11) to move every Hochschild dot to that edge.
(3) If there are two or more Hochschild dots, then the diagram equals the zero morphism by (4.5). If there is exactly one Hochschild dot, then use (4.10) to "pop it out" to a region bordered by the top boundary.

This procedure turns every diagram relevant to the present lemma into a $\Bbbk$-linear combination of diagrams of the following types, depending on if it contains a Hochschild dot or not. In each diagram below, x is some homogeneous element in $\in \Lambda^\vee$, and the dotted part of the diagram is an ordinary Elias–Khovanov diagram.

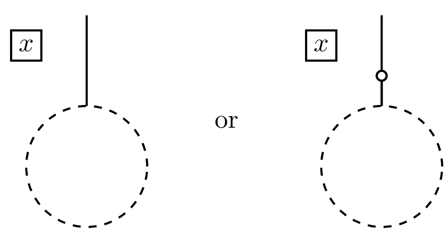
- For $B_\varnothing \to B_\varnothing$:



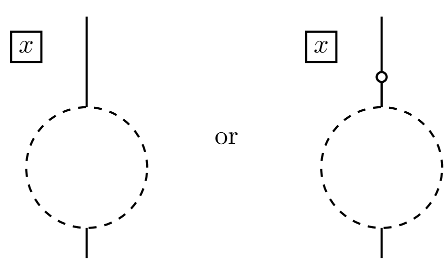
- For $B_s \to B_\varnothing$:



- For $B_\varnothing \to B_s$:



- For $B_s \to B_s$:



Every equality now follows from Lemma 4.6 and the Hochschild barbell relation (4.4). The last statement of the lemma follows from (4.12) and (4.15). $\square$

5. From diagrammatics to bimodules

Now come back to the setting of §3. The following theorem is essentially due to Elias–Khovanov [EK].

Theorem 5.1. *There is a* $\Bbbk$*-linear monoidal equivalence*

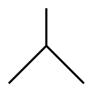
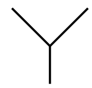
$$\mathcal{F} : \mathscr{D}(\mathfrak{h}, S_2) \to \mathrm{BSBim}(\mathfrak{h}, S_2)$$

determined on objects by

$$\mathcal{F}(B_s) = B_s^{\mathrm{bim}}$$

and on morphisms by

$\mathcal{F}($  $) = \epsilon_s^{\mathrm{bim}}$, $\mathcal{F}($  $) = \eta_s^{\mathrm{bim}}$

$\mathcal{F}($  $) = m_s^{\mathrm{bim}}$, $\mathcal{F}($  $) = c_s^{\mathrm{bim}}$

$$\mathcal{F}(\ \boxed{f}\ ) = m_f^{\mathrm{bim}} \quad (f \in R \textit{ homogeneous}).$$

The following is the first main result of this paper.

Theorem 5.2. *There exists a* $\Bbbk$*-linear monoidal functor*

$$\mathcal{F}^{\mathrm{Ext}} : \mathscr{D}^{\mathrm{Ext}}(\mathfrak{h}, S_2) \to \mathrm{BSBim}^{\mathrm{Ext}}(\mathfrak{h}, S_2)$$

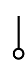
extending the functor $\mathcal{F}$ *of Theorem 5.1 and satisfying*

$\mathcal{F}^{\mathrm{Ext}}($  $) = \phi_s$, $\quad \mathcal{F}(\ \boxed{x}\ ) = \iota_x \quad$ *for* $x \in \Lambda^\vee$ *homogeneous,*

where ϕ_s *and* ι_x *are the morphism defined in §3.7.4 and §3.7.2, respectively.*

Proof. We need to show that the images of the generating morphisms satisfy the new relations in §4.1.3.

By Lemma 3.8, we have

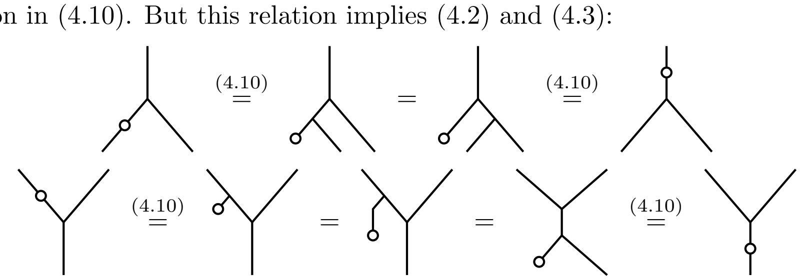
(5.1) $\mathcal{F}^{\mathrm{Ext}}($  $) = \eta_s^{\mathrm{Ext}}$.

Using (5.1), the relation

 $= \boxed{\alpha_s^\vee}$

is easily checked from the explicit description of the chain maps involved. This together with the remaining relations show the rest of the Hochschild barbell relation (4.4).

Also using (5.1), (3.30) says that the images of the generators satisfy the first relation in (4.10). But this relation implies (4.2) and (4.3):



and similarly for the reflection along a vertical axis.

The relation (4.5) holds by degree reason, and (4.6) is clear.

It remains to verify the exterior forcing relation (4.7). By Remark 4.4, the exterior forcing relation follows from that for exterior boxes labeled $x \in V[\![-1]\!] \subset \Lambda^\vee$, and this linear version was checked in (3.31). $\square$

Theorem 5.3. *The functor $\mathcal{F}^{\mathrm{Ext}}$ of Theorem 5.2 is an equivalence.*

Proof. The functor $\mathcal{F}^{\mathrm{Ext}}$ is essentially surjective by construction.

FINISH Show it is full using the computation from §3.

FINISH Use the computation of morphism spaces in §4.3 to show that $\mathrm{Hom}_{\mathscr{D}^{\mathrm{Ext}}}$ has graded dimension less than or equal to $\mathbb{H}\mathsf{om}$, and conclude that $\mathcal{F}^{\mathrm{Ext}}$ is also faithful. $\square$

References

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New York, NY, U.S.A.
Email address: smakisumi@gmail.com